Viscosity in strongly coupled gauge theories

Lessons from string theory

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A review of many people's work

Basic setup

Relativistic field theories:

In $d = 3{+}1$ dimensions

Translation + rotation invariant

Have stable thermal equilibrium state

For simplicity $\mu = 0$, not near critical point

Small fluctuations on top of thermodynamic equilibrium state:

Slowly relaxing d.o.f.: $\epsilon = T^{00}$, $\pi^i = T^{0i} = \langle \epsilon{+}P \rangle v^i$, $\rho = J^0$

Obey classical equations — hydrodynamics

Does not depend on microscopic details of the theory

Hydrodynamic fluctuations

- Conservation laws: $\partial_\mu T^{\mu\nu}=0 \quad \Rightarrow \quad \begin{cases} \partial_t \epsilon = -\boldsymbol{\nabla}\cdot\boldsymbol{\pi} \\ \partial_t \pi^i = -\boldsymbol{\nabla}_j T^{ij} \end{cases}$

- Constitutive relations:

$$\begin{cases} T^{ij} = \delta^{ij}\left[\langle P\rangle + v_s^2\delta\epsilon - \gamma_\zeta \boldsymbol{\nabla}\boldsymbol{\pi}\right] - \gamma_\eta\left(\boldsymbol{\nabla}^i\pi^j + \boldsymbol{\nabla}^j\pi^i - \frac{2}{3}\delta^{ij}\boldsymbol{\nabla}\boldsymbol{\pi}\right) + \ldots \\ \gamma_\eta \equiv \dfrac{\eta}{\langle \epsilon+P\rangle},\ \gamma_\zeta \equiv \dfrac{\zeta}{\langle \epsilon+P\rangle},\ v_{\mathrm{s}}^2 = \partial P/\partial\epsilon \end{cases}$$

- Viscosities η, ζ — input from microscopic physics

Two eigenmodes:

Shear mode: $\pi_\perp(t,\boldsymbol{k}) = e^{-\gamma_\eta \boldsymbol{k}^2 t}\,\pi_\perp(0,\boldsymbol{k})$

Sound mode: $\boldsymbol{\pi}_\parallel(t,\boldsymbol{k}) = e^{-\frac{1}{2}(\gamma_\zeta+\frac{4}{3}\gamma_\eta)\boldsymbol{k}^2 t}\times$

$$\times\left[\boldsymbol{\pi}_\parallel(0,\boldsymbol{k})\cos(kv_st) - i\hat{k}v_s\sin(kv_st)\delta\epsilon(0,\boldsymbol{k})\right]$$

Long-wavelength response is controlled by a small number of kinetic coefficients

Correlation functions in the hydrodynamic limit

Hydrodynamic modes $\Rightarrow$ hydrodynamic singularities at small ω, k.

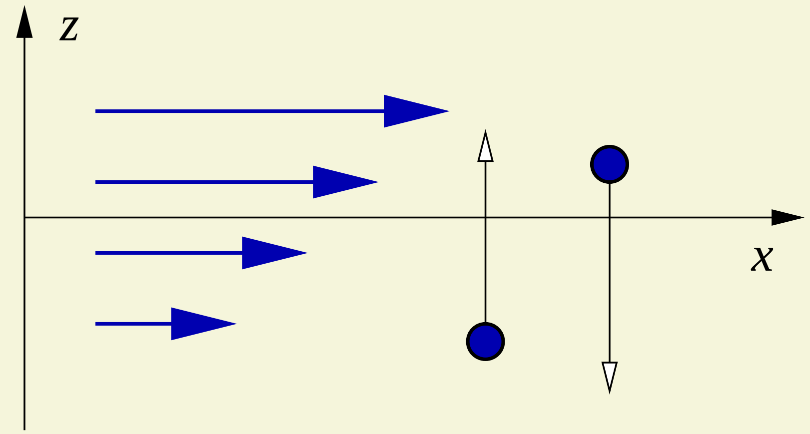
Example: $S_{tx,tx}(\omega,k) = \dfrac{2\gamma_\eta k^2}{\omega^2 + (\gamma_\eta k^2)^2}(\epsilon{+}P)T$ relaxation of transverse momentum

Kubo formulas $$\boxed{\eta = \lim_{\omega\to 0} \frac{1}{2\omega} \int dt\, e^{i\omega t} \int d^3x \, \langle [T_{xy}(t,\boldsymbol{x}), T_{xy}(0)] \rangle}$$

Connection to microscopic physics: Viscosities can be extracted from (small-frequency limits of) real-time correlation functions

Physical meaning of viscosity

Example: flow of gas



Internal friction — momentum exchange between layers of fluid

$T_{zx} = \eta \dfrac{\partial v}{\partial z}$ flux of x-momentum in z-direction

Particle number per unit surface per unit time: $(n v_{\rm th})$

Momentum per unit surface per unit time: $(n v_{\rm th})(mv)$

Flux up: $n v_{\rm th} m (v_0 - l_{\rm mfp} \frac{\partial v}{\partial z})$, flux down: $n v_{\rm th} m (v_0 + l_{\rm mfp} \frac{\partial v}{\partial z})$,

$T_{zx} = \mathrm{flux}_{\rm up} - \mathrm{flux}_{\rm down}$

$$\boxed{\eta \sim n v_{\rm th} m l_{\rm mfp} \sim \frac{m v_{\rm th}}{\sigma} \sim \frac{(mT)^{1/2}}{\sigma}}$$

Ideal gas has infinite viscosity

How viscosity is measured

WEAK COUPLING

Prof. J.Maxwell, Mrs.Maxwell, (attic, 1860)

STRONG COUPLING

RHIC, many people (Brookhaven, 2000)

Strong coupling is more difficult

How viscosity is computed

- Gases: Kinetic theory

$$\boxed{\eta \propto (mT)^{1/2}\,\frac{1}{\sigma}}$$

Density-independent, grows with T

- Plasmas: Boltzmann-Vlasov equation

$$\boxed{\eta \propto (mT)^{1/2}\left(\frac{T^2}{e^4 \ln \Lambda}\right)}$$

"Coulomb log" $\Lambda \sim \frac{l_d T}{e^2} \sim \frac{T^{3/2}}{e^3 n^{1/2}}$ due to small-angle scatt.

- Liquids: Complicated: use molecular dynamics

$$\boxed{\eta \propto \exp\left(\frac{E_a}{T}\right)}$$

Typical: decreases with T

- Weakly coupled QFT: Boltzmann-Vlasov equation, or resum Feynman diagrams

$$\boxed{\eta \propto \frac{T^3}{g^4 \ln g^{-1}}}$$

(Log absent if no gauge fields)

- Non-abelian gauge theories: at $T \gtrsim T_c$ need lattice

Evaluate Euclidean correlators, invert to find real-time spectral functions...

$$\boxed{\eta = ???}$$

Q: Are there any toy models where viscosity can be analytically computed at strong coupling?

A: $\boxed{\mathcal{N}{=}4 \text{ supersymmetric Yang-Mills theory}, SU(N_c), \lambda = g^2 N_c}$

Why $\mathcal{N}{=}4$ SYM?

Dual string description (AdS/CFT correspondence) — allows to compute correlation functions at strong coupling

What is $\mathcal{N}{=}4$ SYM?

- Gauge fields + 4 fermions + 6 scalars in adjoint of $SU(N_c)$
- Conformal theory, λ is a tunable parameter (does not run)
- Supersymmetric, but SUSY not essential at finite temperature
- $\epsilon = 3P$, $v_s = \frac{1}{\sqrt{3}}$, $\zeta = 0$, at any non-zero temperature
- ϵ, P, η are finite in the limit $\lambda \to \infty$

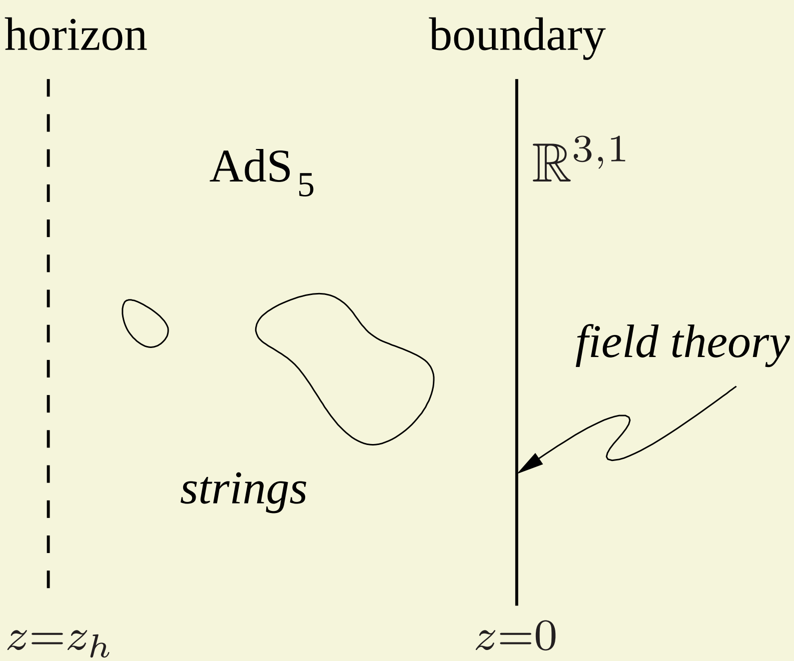
AdS/CFT correspondence

(J.Maldacena `hep-th/9711200`, review: `hep-th/9905111`)

$$\boxed{\text{large } N_c,\ d{=}4,\ \mathcal{N}{=}4 \text{ SYM} \ = \ \text{IIB strings on } AdS_5 \times S^5}$$

$\lambda \leftrightarrow \left(\frac{R^2}{\alpha'}\right)^2$ string corrections to SUGRA

$\frac{\lambda}{4\pi N_c} \leftrightarrow g_s$ string loops



$$\langle e^{\int h(x) T(x)} \rangle_{\text{field}} = Z_{\text{string}}[g(x, z{\to}0) = h(x)]$$

$$T_{\mu\nu}(x) \quad \leftrightarrow \quad h_{\mu\nu}(x, z{\to}0)$$

$$J_\mu(x) \quad \leftrightarrow \quad A_\mu(x, z{\to}0)$$

$$\mathrm{tr} F^2(x) \quad \leftrightarrow \quad \varphi(x, z{\to}0)$$

$$\vdots$$

$$\therefore \ \langle T_{\mu\nu} T_{\alpha\beta} \rangle \sim \frac{\delta^2 \ln Z_{\text{string}}[h]}{\delta h_{\mu\nu} \delta h_{\alpha\beta}} \sim \frac{\delta^2}{\delta h_{\mu\nu} \delta h_{\alpha\beta}} S_{\text{cl}}[h]$$

Duality unproven, but many consistency checks performed

How to compute viscosity from AdS/CFT

(G.Policastro, D.Son, A.Starinets hep-th/0104066, hep-th/0205052,
PK, D.Son, A.Starinets hep-th/0405231, PK, A.Starinets hep-th/0506184)

$$S_{\rm cl} = \int d^4x\, dz \sqrt{-g}\, g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi = \int d^4x \sqrt{-g}\, g^{zz} \varphi(x,z) \partial_z \varphi(x,z) \Big|_{z\to 0}$$

$$\boxed{G^{\rm ret}(k) \sim \frac{1}{z^3} f(k,z) \partial_z f(-k,z) \Big|_{z\to 0}}$$

$f(k,z)$ satisfies e.o.m.

$f(k,z) \to 1$ as $z \to 0$

$f(k,z)$ outgoing as $z \to z_h$

(breaks time-reversal invariance)

– For $G^{\rm ret}_{\mu\nu,\alpha\beta}(k)$, need linearized metric perturbations $h_{\rho\lambda}$ $\Rightarrow$ coupled ODEs

– Shear viscosity $\eta \sim \langle T_{xy} T_{xy} \rangle$, need $h_{xy}(x,z)$

– $h^y_x = \varphi$ (minimal massless scalar), and decouples (true more generally)

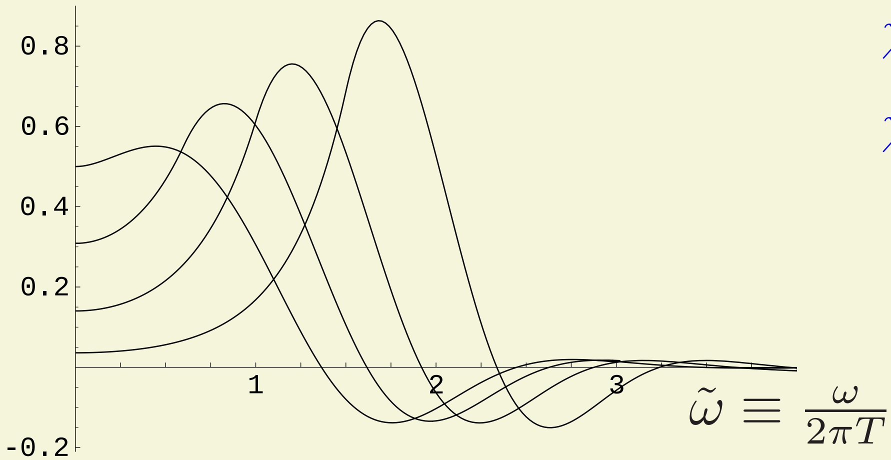
– In the hydro limit $\omega/T \ll 1$, $|\boldsymbol{k}|/T \ll 1$, can solve analytically

Computing real-time correlation functions in strongly coupled SYM amounts to solving a wave equation on a background with BH

Spectral function for stress

(PK, A.Starinets hep-th/0602059, D.Teaney hep-ph/0602044)

$$\frac{1}{\tilde{\omega}}\left(\chi(\tilde{\omega}) - \chi^{T=0}(\tilde{\omega})\right) \ \left[\tfrac{1}{\pi^2 N_c^2 T^4}\right]$$



$$\chi(\omega, k) = -2\,\mathrm{Im}\,G^{\mathrm{ret}}_{xy,xy}(\omega, k)$$

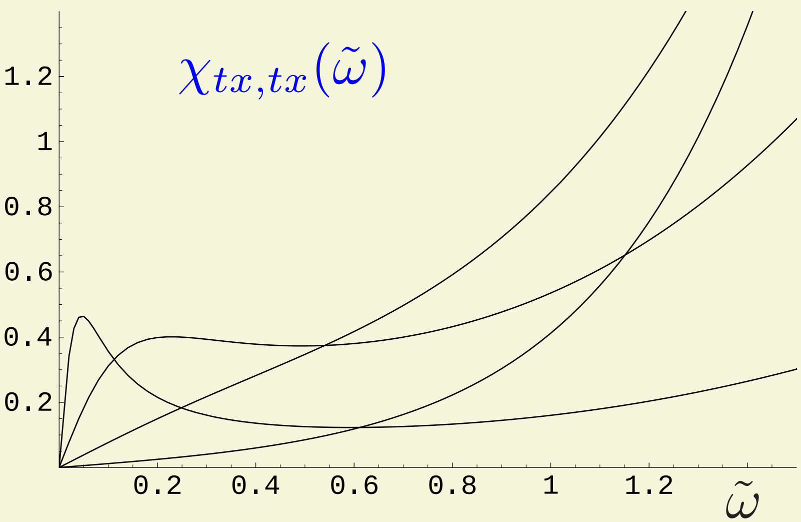
$$\chi(\omega) \sim \omega, \quad \omega \ll 2\pi T$$

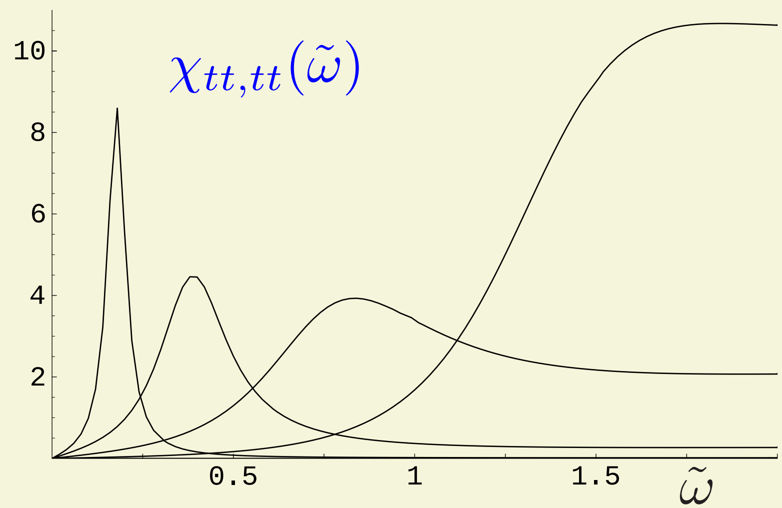
$$\chi(\omega) - \chi^{T=0}(\omega) \sim e^{-\gamma\omega}, \quad \omega \gg 2\pi T$$

$$\boxed{\eta = \tfrac{\pi}{8} N_c^2 T^3}$$

T^3 by conformal invariance, N_c^2 counts d.o.f.

Spectral function for conserved energy-momentum



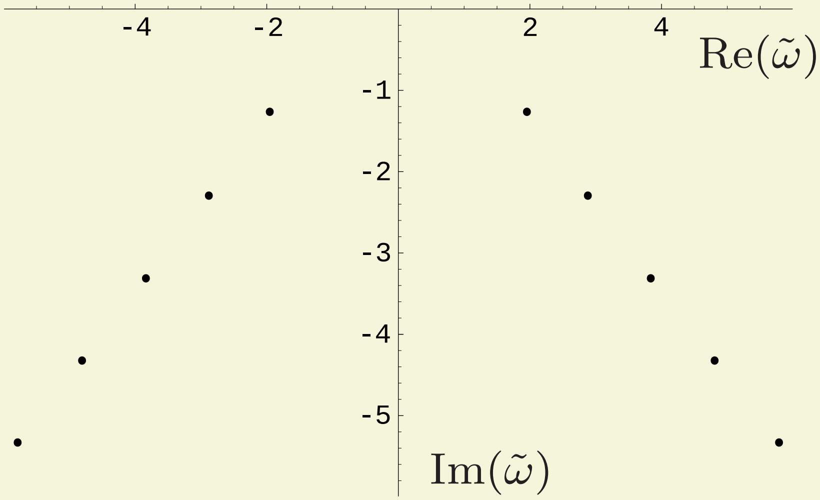


Hydrodynamic peaks clearly visible in dual classical gravity

Singularities of $G^{\rm ret}(\omega, k)$

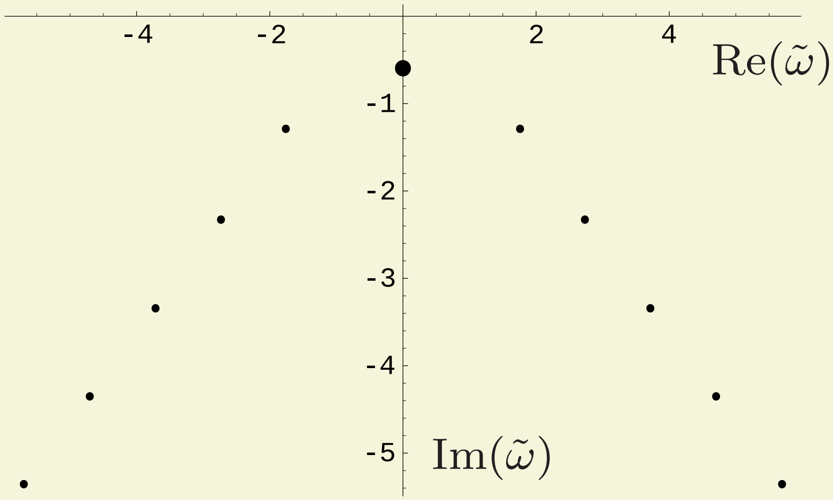
(A.Nunez, A.Starinets hep-th/0302026, PK, A.Starinets hep-th/0506184)

$G^{\rm ret}_{xy,xy}(\omega)$

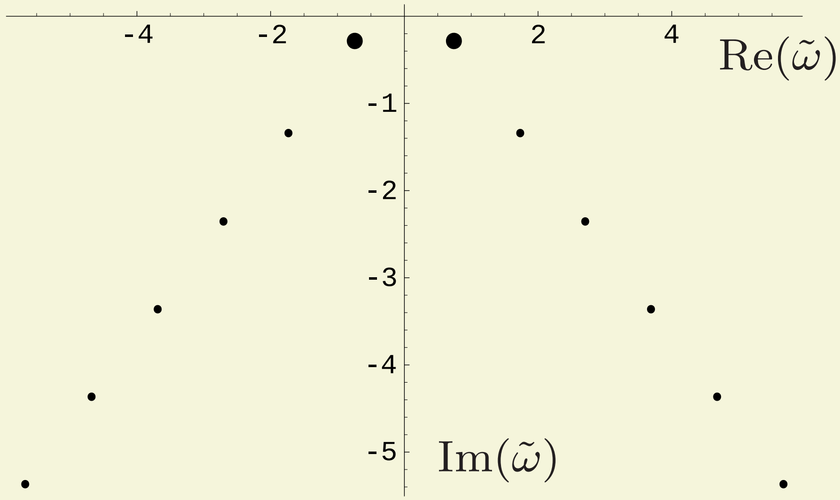


- Infinite series of poles
- $\omega_n = 2\pi n T(\pm 1 - i)$ as $n \to \infty$
- For conserved densities, $\omega_0 \to 0$ as $k \to 0$
- Hydro poles agree with Kubo formula

$G^{\rm ret}_{tx,tx}(\omega)$



$G^{\rm ret}_{tt,tt}(\omega)$

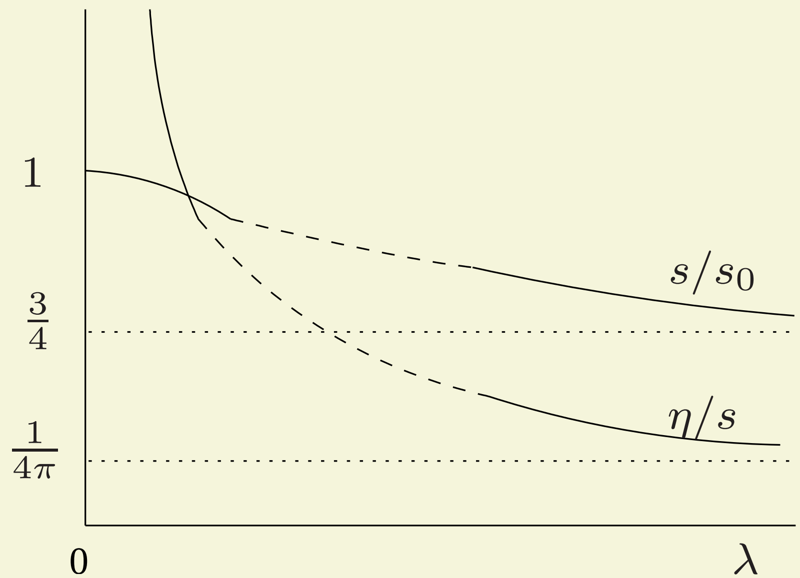


Singularities of $G^{\rm ret}(\omega, k)$ are (quasi)normal modes of the dual gravity background

Universality of η/s

(PK, D.Son, A.Starinets hep-th/0405231, A.Buchel hep-th/0408095)

- Hydro damping rate is set by η/s
- Shear viscosity $\eta \sim N_c^2$, but η/s is finite in the $N_c \to \infty$ limit



At strong coupling:

$$s = \tfrac{3}{4} s(\lambda{=}0)\,[1 + O(\lambda^{-3/2})]$$
$$\eta/s = \tfrac{1}{4\pi}\,[1 + O(\lambda^{-3/2})]$$

$$\boxed{\frac{\eta}{s} = \frac{1}{4\pi} \text{ does not depend on the dual gravity background !!!}}$$

Conformal invariance, SUSY, 3+1 dimensions — not essential for the universality

$\therefore\ \eta/s = \frac{1}{4\pi}$ for a large class of strongly coupled field theories

Speculation: Lower bound on shear viscosity?

Life at low Reynolds number

E. M. Purcell
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(Received 12 June 1976)

Editor's note: This is a reprint (slightly edited) of a paper of the same title that appeared in the book *Physics and Our World: A Symposium in Honor of Victor F. Weisskopf*, published by the American Institute of Physics (1976). The personal tone of the original talk has been preserved in the paper, which was itself a slightly edited transcript of a tape. The figures reproduce transparencies used in the talk. The demonstration involved a tall rectangular transparent vessel of corn syrup, projected by an overhead projector turned on its side. Some essential hand waving could not be reproduced.

This is a talk that I would not, I'm afraid, have the nerve to give under any other circumstances. It's a story I've been saving up to tell Viki. Like so many of you here, I've enjoyed from time to time the wonderful experience of exploring with Viki some part of physics, or anything to which we can apply physics. We wander around strictly as amateurs equipped only with some elementary physics, and in the end, it turns out, we improve our understanding of the elementary physics even if we don't throw much light on the other subjects. Now this is that kind of a subject, but I have still another reason for wanting to, as it were, needle Viki with it, because I'm going to talk for a while about viscosity. Viscosity in a liquid will be the dominant theme here and you know Viki's program of explaining everything, including the heights of mountains, with the elementary constants. The viscosity of a liquid is a very tough nut to crack, as he well knows, because when the stuff is cooled by merely 40 degrees, its viscosity can change by a factor of a million. I was really amazed by fluid viscosity in the early days of NMR, when it turned out that glycerine was just what we needed to explore the behavior of spin relaxation. And yet if you were a little bug inside the glycerine, looking around, you wouldn't see much change in your surroundings as the glycerine cooled. Viki will say that he can at least predict the *logarithm* of the viscosity. And that, of course, is correct because the reason viscosity changes is that it's got one of these activation energy things and what he can predict is the order of magnitude of the exponent. But it's more mysterious than that, Viki, because if you look at the Chemical Rubber Handbook table you will find that there is almost no liquid with viscosity much lower than that of water. The viscosities have a big range *but they stop at the same place*. I don't understand that. That's what I'm leaving for him.[1]

Now, I'm going to talk about a world which, as physicists, we almost never think about. The physicist hears about viscosity in high school when he's repeating Millikan's oil drop experiment and he never hears about it again, at least not in what I teach. And Reynolds's number, of course, is something for the engineers. And the *low* Reynolds number regime most engineers aren't even interested in—except possibly chemical engineers, in connection with fluidized beds, a fascinating topic I heard about from a chemical engineering friend at MIT. But I want to take you into the world of very low Reynolds number—a world which is inhabited by the overwhelming majority of the organisms in this room. This world is quite different from the one that we have developed our intuitions in.

I might say what got me into this. To introduce something that will come later, I'm going to talk partly about how microorganisms swim. That will not, however, turn out to be the only important question about them. I got into this through the work of a former colleague of mine at Harvard, Howard Berg. Berg got his Ph.D. with Norman Ramsey, working on a hydrogen maser, and then he went back into biology which had been his early love, and into cellular physiology. He is now at the University of Colorado at Boulder, and has recently participated in what seems to me one of the most astonishing discoveries about the questions we're going to talk about. So it was partly Howard's work, tracking *E. coli* and finding out this strange thing about them, that got me thinking about this elementary physics stuff.

Well, here we go. In Fig. 1, you see an object which is moving through a fluid with velocity v. It has dimension a. In Stokes's law, the object is a sphere, but here it's anything; η and ρ are the viscosity and density of the fluid. The ratio of the inertial forces to the viscous forces, as Osborne Reynolds pointed out slightly less than a hundred years ago, is given by $av\rho/\eta$ or av/ν, where ν is called the *kinematic* viscosity. It's easier to remember its dimensions: for water, $\nu \approx 10^{-2}\ \mathrm{cm^2/sec}$. The ratio is called the Reynolds number and when that number is small the viscous forces dominate. Now there is an easy way, which I didn't realize at first, to see who should be interested in small Reynolds numbers. If you take the viscosity η and square it and divide by the density, you get a force (Fig. 2). No other dimensions come in at all. η^2/ρ is a force. For water, since $\eta \approx 10^{-2}$ and $\rho \approx 1$, $\eta^2/\rho \approx 10^{-4}$ dyn. That is a force that will tow *anything*, large or small, with a Reynolds number of order of magnitude 1. In other words, if you want to tow a submarine with Reynolds number 1 (or strictly speaking, $1/6\pi$ if it's a spherical submarine) tow it with 10^{-4} dyn. So it's clear in this case that you're interested in small Reynolds number if you're interested in *small forces* in an absolute sense. The only other people who are interested in low Reynolds number, although they usually don't have to invoke it, are the geophysicists. The Earth's mantle is supposed to have a viscosity of 10^{21} P. If you now work out η^2/ρ, the force is 10^{41} dyn. That is more than 10^9 times the gravitational force that half the Earth exerts on the other half! So the conclusion is, of course, that in the flow of the mantle of the Earth the Reynolds number is *very* small indeed.

Now consider things that move through a liquid (Fig. 3). The Reynolds number for a man swimming in water might be 10^4, if we put in reasonable dimensions. For a goldfish or a tiny guppy it might get down to 10^2. For the animals that we're going to be talking about, as we'll see in a mo-

he can predict is the order of magnitude of the exponent. But it's more mysterious than that, Viki, because if you look at the Chemical Rubber Handbook table you will find that there is almost no liquid with viscosity much lower than that of water. The viscosities have a big range *but they stop at the same place*. I don't understand that. That's what I'm leaving for him.[1]

!!

- $\eta/s \gg 1$ at small coupling
- η/s is finite at large coupling
- Natural to assume $\eta/s \geqslant \frac{1}{4\pi}$ in SYM

$$\boxed{\text{Is } \frac{\eta}{s} \geqslant \frac{1}{4\pi} \text{ universal?}}$$

Prove from first principles?

Generalizations

- Correction $\eta/s = \frac{1}{4\pi}[1 + O(\frac{1}{\lambda^{3/2}})]$ known explicitly in $\mathcal{N}$=4 SYM

  (A.Buchel, J.Liu, A.Starinets hep-th/0406264)

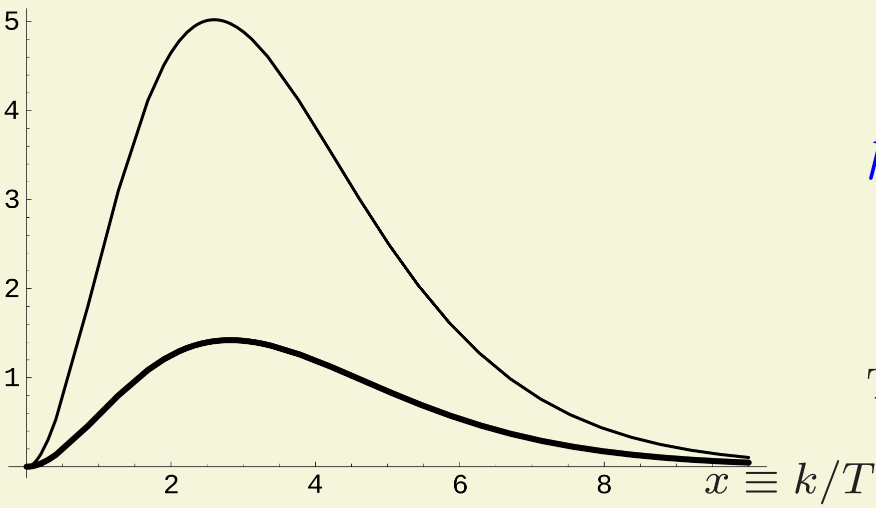
- Add mass terms to $\mathcal{N}$=4 SYM (destroys scale invariance, reduces SUSY):
  $\eta/s = \frac{1}{4\pi}$, $\zeta/\eta \approx -4.56(v_{\mathrm{s}}^2 - \frac{1}{3})$ (P.Benincasa, A.Buchel, A.Starinets hep-th/0507026)

- Non-zero chemical potential: $\eta/s = \frac{1}{4\pi}$ in $\mathcal{N}$=4 SYM, does not depend on $\frac{\mu}{T}$

  (D.Son, A.Starinets hep-th/0601157, J.Mas hep-th/0601144, K.Maeda, M.Natsuume, T.Okamura hep-th/0602010)

Photon production from SYM

(PK, A.Starinets, to appear)



$$k\frac{d\Gamma}{dk} \propto \frac{k^2\,\eta^{\mu\nu}}{e^{k/T}-1}\,\mathrm{Im}\,\Pi^{\mathrm{ret}}_{\mu\nu}(\omega,k)\Big|_{\omega^2=k^2}$$

Thick line: $\dfrac{x^3}{e^x-1}$ Thin line: $\dfrac{x^3}{e^x-1}f_{\mathrm{SYM}}(x)$

At small x: $f_{\mathrm{SYM}}(x) \to \mathrm{const}$, $\mathrm{Im}\,\Pi^{\mu}{}_{\mu}(\omega{=}k) \sim k$, consistent with hydrodynamics

Summary

- Viscosity can be relatively easily computed for some strongly coupled theories such as $\mathcal{N}=4$ SYM
- Ratio of viscosity to entropy density is $1/4\pi$ for a large class of strongly coupled gauge theories. Lower bound $\eta/s \geqslant \frac{1}{4\pi}$?
- Limitations of the AdS/CFT approach:
  - SUSY theories, contain scalar fields
  - Conformal in the UV rather than AF
  - Fundamental matter $N_\mathrm{f} \ll N_\mathrm{c}$
  - $1/\lambda$ corrections doable, $1/N_\mathrm{c}$ corrections hard
- Search for universal properties of strongly interacting thermal gauge theories
- Other questions can be addressed in AdS/CFT: energy loss by a heavy quark, existence of resonances, photon production, thermalization...

Can AdS/CFT be useful for heavy-ion physics?