<table>
<thead>
<tr>
<th>Nature of the transition</th>
<th>Transition temperature</th>
<th>Equation of state</th>
<th>Curvature on $\mu-T$</th>
<th>Summary</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Z. Fodor</td>
</tr>
</tbody>
</table>

QCD thermodynamics from the lattice: Continuum limit with...
QCD thermodynamics from the lattice: Continuum limit with physical quark masses

Z. Fodor

University of Wuppertal, Eotvos University Budapest, Forschungszentrum Juelich

results of the Wuppertal-Budapest group: Nature, 443 ’06 675, JHEP 1009 ’10 73, 1011 ’10 77, 1102.1356

February 9, 2011, Winter Park, Colorado
Outline

1. Nature of the transition
2. Transition temperature
3. Equation of state
4. Curvature on $\mu-T$
5. Summary
User’s guide to lattice QCD results

- Full lattice results have three main ingredients

1. (tech.) technically correct (users can not really prove)
2. \((m_q)\) physical quark masses: \(m_s/m_{ud} \approx 28\) (and \(m_c/m_s \approx 12\))
3. (cont.) continuum extrapolated: at least 3 points with \(c \cdot a^n\)

only a few full results (spectrum, \(m_q\), nature, \(T_c\), EoS, curvature)

ad 1: obvious condition, otherwise forget it
ad 2: difficult (CPU demanding) to reach the physical u/d mass
BUT even with non-physical quark masses: meaningful questions
e.g. in a world with \(M_\pi = M_\rho\) what would be \(M_N/M_\pi\)
these results are universal, do not depend on the action/technique
ad 3: non-continuum results contain lattice artefacts
(they are good for methodological studies, they just "inform" you)
User’s guide to lattice QCD results

- troubleshooting
  i. clarify if all three conditions were satisfied
  ii. if yes: OK with any scale setting (error estimates can be tricky)
  iii. if out of the above three ingredients one was missing:
    - if (1: tech.) was missing: forget it
    - if (2: $m_q$) was missing: reliable answer to a well defined case
    - if (3: cont.) was missing: ask to carry out the continuum limit show the scaling $c \cdot a^n$ in the scaling regime
      n is known from theory c is provided by the simulations
      that is why we need at least 3 different lattice spacings
  iv. if out of the above three ingredients two were missing: well ...
physical quark masses: important for the nature of the transition

\( n_f=2+1 \) theory with \( m_q=0 \) or \( \infty \) gives a first order transition

intermediate quark masses: we have an analytic cross over (no \( \chi PT \))

F.Karsch et al., Nucl.Phys.Proc. 129 ('04) 614; G.Endrodi et al. PoS Lat'07 182('07);

de Forcrand, S. Kim, O. Philipsen, Lat'07 178('07)

continuum limit is important for the order of the transition:

\( n_f=3 \) case (standard action, \( N_t=4 \)): critical \( m_{ps} \approx 300 \) MeV

different discretization error (p4 action, \( N_t=4 \)): critical \( m_{ps} \approx 70 \) MeV

the physical pseudoscalar mass is just between these two values
Finite size scaling in the quenched theory

look at the susceptibility of the Polyakov-line first order transition (Binder) $\Rightarrow$ peak width $\propto 1/V$, peak height $\propto V$

finite size scaling shows: the transition is of first order
Approaching the continuum limit

\[ a = 0.3 \text{ fm} \]

3.6 fm 4.8 fm 6 fm

\[ \frac{1}{N_t^2} \propto a^2 \]

\[ \frac{1}{N_t^2} \propto a^2 \]

\[ \frac{1}{N_t^2} \propto a^2 \]
Approaching the continuum limit

\[ a = 0.2 \text{ fm} \]

3.6 fm  4.8 fm  6 fm

\[ \frac{1}{N_t^2} \propto a^2 \]

\[ \frac{N_s}{N_t} = 3 \]
\[ \frac{N_s}{N_t} = 4 \]
\[ \frac{N_s}{N_t} = 5 \]
Approaching the continuum limit

\[ a = 0.15 \text{ fm} \]

\[ 3.6 \text{ fm} \quad 4.8 \text{ fm} \quad 6 \text{ fm} \]

\[ \frac{T^4}{m^2 \lambda^4} \]

\[ \frac{1}{N_t^2} \propto a^2 \]

\[ \frac{1}{N_t^2} \propto a^2 \]

\[ \frac{1}{N_t^2} \propto a^2 \]
Approaching the continuum limit

\[ a = 0.12 \text{ fm} \]

3.6 fm  4.8 fm  6 fm

\[
\frac{T^4}{m^2 a^4} \\
\frac{1}{N_t^2} \propto a^2
\]

\[
\frac{N_s}{N_t} = 3 \quad \frac{N_s}{N_t} = 4 \quad \frac{N_s}{N_t} = 5
\]
### Approaching the continuum limit

<table>
<thead>
<tr>
<th>Nature of the transition</th>
<th>Transition temperature</th>
<th>Equation of state</th>
<th>Curvature on $\mu$–$T$</th>
<th>Summary</th>
</tr>
</thead>
</table>

Approaching the continuum limit

3.6 fm  
4.8 fm  
6 fm

![Graph showing $T^4/(m^2\alpha^2)$ vs. $1/N_t^2 \propto \alpha^2$ for $N_s/N_t = 3$, $N_s/N_t = 4$, and $N_s/N_t = 5$.](image_url)
The nature of the QCD transition


analytic transition (cross-over) ⇒ it has no unique $T_c$:
examples: melting of butter (not ice) & water-steam transition

above the critical point $c_p$ and $d\rho/dT$ give different $T_c$s.

QCD: chiral & quark number susceptibilities or Polyakov loop
they result in different $T_c$ values ⇒ physical difference
Literature: discrepancies between $T_c$

Bielefeld-Brookhaven-Riken-Columbia Collaboration:

$T_c$ from $\chi_{\bar{\psi}\psi}$ and Polyakov loop, from both quantities:

$T_c = 192(7)(4)$ MeV

Bielefeld-Brookhaven-Riken-Columbia merged with MILC: ‘hotQCD’

Wuppertal-Budapest group: WB

chiral susceptibility:

$T_c = 151(3)(3)$ MeV

Polyakov and strange susceptibility:

$T_c = 175(2)(4)$ MeV

‘chiral $T_c$’: $\approx 40$ MeV; ‘confinement $T_c$’: $\approx 15$ MeV difference

both groups give continuum extrapolated results with physical $m_\pi$
Chiral symmetry breaking and pions

transition temperature for remnant of the chiral transition:
balance between the f’s of the chirally broken & symmetric sectors
chiral symmetry breaking: 3 pions are the pseudo-Goldstone bosons

staggered QCD: 1 (3/16) pseudo-Goldstone instead of 3 (taste violation)
staggered lattice artefact ⇒ disappears in the continuum limit
WB: stout-smeared improvement is designed to reduce this artefact
progress in the transition temperature

Wuppertal-Budapest: physical quark masses \( m_s/m_{ud} \approx 28 \)
gauge configs: \( N_t=8,10 \) in 2006 \( \Rightarrow \) \( N_t=12 \) in 2009 \( \Rightarrow \) \( N_t=16 \) in 2010

hotQCD 2009: realistic quark masses \( m_s/m_{ud} = 10 \)
hotQCD 2010: preliminary: physical quark masses \( m_s/m_{ud} = 20 \)
progress in the transition temperature

Wuppertal-Budapest: physical quark masses \( \left( \frac{m_s}{m_{ud}} \approx 28 \right) \)
gauge configs: \( N_t = 8, 10 \) in 2006 \( \Rightarrow \) \( N_t = 12 \) in 2009 \( \Rightarrow \) \( N_t = 16 \) in 2010

hotQCD 2009: realistic quark masses \( \left( \frac{m_s}{m_{ud}} = 10 \right) \)
hotQCD 2010: preliminary: physical quark masses \( \left( \frac{m_s}{m_{ud}} = 20 \right) \)
progress in the transition temperature

Wuppertal-Budapest: physical quark masses \((m_s/m_{ud} \approx 28)\)
gauge configs: \(N_t=8,10\) in 2006 \(\Rightarrow\) \(N_t=12\) in 2009 \(\Rightarrow\) \(N_t=16\) in 2010

hotQCD 2009: realistic quark masses \((m_s/m_{ud} = 10)\)
hotQCD 2010: preliminary: physical quark masses \((m_s/m_{ud} = 20)\)
progress in the transition temperature

Wuppertal-Budapest: physical quark masses \((m_s/m_{ud} \approx 28)\)
gauge configs: \(N_t=8,10\) in 2006 \(\Rightarrow\) \(N_t=12\) in 2009 \(\Rightarrow\) \(N_t=16\) in 2010

hotQCD 2009: realistic quark masses \((m_s/m_{ud} = 10)\)
hotQCD 2010: preliminary: physical quark masses \((m_s/m_{ud} = 20)\)
**progress in the transition temperature**

Wuppertal-Budapest: physical quark masses \( (m_s/m_{ud} \approx 28) \)
gauge configs: \( N_t=8,10 \) in 2006 \( \Rightarrow \) \( N_t=12 \) in 2009 \( \Rightarrow \) \( N_t=16 \) in 2010

hotQCD 2009: realistic quark masses \( (m_s/m_{ud} =10) \)
hotQCD 2010: preliminary: physical quark masses \( (m_s/m_{ud} =20) \)
progress in the transition temperature

Wuppertal-Budapest: physical quark masses \( (m_s/m_{ud} \approx 28) \)
gauge configs: \( N_t=8,10 \) in 2006 \( \Rightarrow N_t=12 \) in 2009 \( \Rightarrow N_t=16 \) in 2010

hotQCD 2009: realistic quark masses \( (m_s/m_{ud} =10) \)
hotQCD 2010: preliminary: physical quark masses \( (m_s/m_{ud} =20) \)
progress in transition temperature

Wuppertal-Budapest: physical quark masses \((m_s/m_{ud} \approx 28)\)
gauge configs: \(N_t=8,10\) in 2006 \(\Rightarrow N_t=12\) in 2009 \(\Rightarrow N_t=16\) in 2010

hotQCD 2009: realistic quark masses \((m_s/m_{ud} = 10)\)
hotQCD 2010: preliminary: physical quark masses \((m_s/m_{ud} = 20)\)
temperature dependence of the chiral condensate

Wuppertal-Budapest: good agreement with the physical HRG


hotQCD: agreement only with the distorted spectrum though their results are gradually getting closer to ours
Quark number susceptibility and baryon-strangeness correlation:

\[
\chi_{ff'}^{\mu\mu} = \frac{T}{V} \frac{\partial^2 \ln Z}{\partial \mu_f \partial \mu_{f'}} \bigg|_{\mu_i=0}
\]

\[
C_{BS} = -3 \frac{\langle BS \rangle}{\langle S^2 \rangle} = 1 + \frac{\chi_{2}^{us} + \chi_{2}^{ds}}{\chi_{2}^{ss}}
\]

\(\chi\) for light and strange quarks show differ up to \(T \sim 2T_c\)

\(C_{BS} \rightarrow 1\) very quickly above \(T_c\): good "order parameter"

for phenomenologists: it supports the picture of independent quarks
Equation of state: integral method


on the lattice the dimensionless pressure is given by

\[ p_{\text{lat}}(\beta, m_q) = (N_t N_s^3)^{-1} \log Z(\beta, m_q) \]

not accessible using conventional algorithms, only its derivatives

\[ p_{\text{lat}}(\beta, m_q) - p_{\text{lat}}(\beta^0, m_q^0) = (N_t N_s^3)^{-1} \int_{(\beta^0, m_q^0)}^{(\beta, m_q)} \left( d\beta \frac{\partial \log Z}{\partial \beta} + dm_q \frac{\partial \log Z}{\partial m_q} \right) \]

first term: gauge action & second term: chiral condensate

the pressure has to be renormalized: subtraction at \( T=0 \) (or \( T>0 \))

\( T \neq 0 \) simulations can’t go below \( T \approx 100 \text{ MeV} \) (lattice spacing is large)

physical HRG gives here 5% contribution of SB ⇒

path of \( M_\pi = 720 \text{ MeV} \) ⇒ distorted HRG no contribution at \( T=100 \text{ MeV} \)
finite \( V \): \( N_s/N_t = 3 \) and 6 (8 times larger volume): no sizable difference

finite \( a \): improvement program of lattice QCD (action & observables)
tree-level improvement for \( p \) (thermodynamic relations fix the others)
trace anomaly for three \( T \)-s: high \( T \), transition \( T \), low \( T \)
continuum limit \( N_t = 6, 8, 10, 12 \): same with or without improvement

improvement strongly reduces cutoff effects: slope \( \approx 0 \) (1-2\( \sigma \) level)
Pressure and energy density

$\epsilon$ normalized to the Stefan-Boltzmann limit: $\epsilon(T \rightarrow \infty) = 15.7$

at 1000 MeV still 20% difference to the Stefan-Boltzmann value

especially perfect scaling, lines/points are lying on top of each other
entropy and trace anomaly

good agreement with the HRG model up to the transition region $T_c$ can be defined as the inflection point of the trace anomaly

<table>
<thead>
<tr>
<th>Inflection point of $I(T)/T^4$</th>
<th>154(4) MeV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$ at the maximum of $I(T)/T^4$</td>
<td>187(5) MeV</td>
</tr>
<tr>
<td>Maximum value of $I(T)/T^4$</td>
<td>4.1(1)</td>
</tr>
</tbody>
</table>

### Speed of sound & parametrization

$c_s$ minimum value is about 0.13 at $T \approx 145$ MeV

’smaller than error’ parametrization $T=100...1000$ MeV ($t=T/200$ MeV)

\[
\frac{l(T)}{T^4} = \exp\left(-\frac{h_1}{t} - \frac{h_2}{t^2}\right) \cdot \left(h_0 + \frac{f_0 \cdot \left[\tanh(f_1 \cdot t + f_2) + 1\right]}{1 + g_1 \cdot t + g_2 \cdot t^2}\right)
\]

<table>
<thead>
<tr>
<th>$h_0$</th>
<th>$h_1$</th>
<th>$h_2$</th>
<th>$f_0$</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$g_1$</th>
<th>$g_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1396</td>
<td>-0.1800</td>
<td>0.0350</td>
<td>2.76</td>
<td>6.79</td>
<td>-5.29</td>
<td>-0.47</td>
<td>1.04</td>
</tr>
</tbody>
</table>

Z. Fodor

QCD thermodynamics from the lattice: Continuum limit with
Equation of state: \( I(T) = \epsilon - 3p \)

two pion masses: \( M_\pi \approx 720 \text{ MeV (} R=1) \) and \( M_\pi = 135 \text{ MeV (} R^{\text{phys}}) \)

good agreement with the HRG model up to the transition region

quark mass dependence disappears for high T

good agreement with perturbation theory

comparison with the published results of the hotQCD collaboration

discrepancy: peak at \( \approx 20 \text{ MeV larger T} \) and \( \approx 50\% \) higher
Charm contribution

perturbative indications: important already at $2 \cdot T_c$


determine it within the partially quenched framework: $m_C/m_s = 11.85$

charm contribution is indeed non-negligible from 200 MeV
one has to extend this observation to the dynamical case
Scenarios for $\mu > 0$

Does the crossover region shrink or expand?

The curvature can affect the existence of the critical endpoint

Estimate: if $\mu_{\text{crit}} = 360$ MeV $\rightarrow \Delta \kappa \approx 0.02$
Equivalence of the methods (formal/numerical)

⇒ for moderate \( \mu \) Taylor and \( \mu_I \) agree with reweighting

take \( n_f=2 \) setting of de Forcrand-Philipsen: \( \beta_c(\mu) \) up to 4 digits

solid/dotted: imaginary \( \mu \) & error; box: reweighting; circle: Taylor

for larger \( \mu \) values higher order terms are getting more important

what to choose (depends on the question):

for this particular case imaginary \( \mu \) has the largest CPU demand;
next one is reweighting; cheapest is Taylor (does not work for large \( \mu \))
all results are from coarse lattices (a=0.3 fm, read our abstract!)
deForcrand-Philipsen: leading order ⇒ not stronger, slightly weaker
same from reweighting (critical point: result of the higher order terms)
Taylor & radius of convergence (!) only a lower bound: Lee-Yang
full answer (all the way to the continuum) needs much more CPU
The curvature

we change $\mu$ and look at the transition curve
it shifts to the left, we look at its value of a fixed $C$

the dimensionless curvature is defined as $\kappa(T) = -T_c(\mu = 0) \cdot R(T)$

$d\kappa/dT$ at $T_c$ tells if the transition is broadening or narrowing
(a point below $T_c$ has a larger or smaller curvature)
Continuum prediction for the curvature: full result

lower solid line: $T_c$ from the chiral condensate
upper solid line: $T_c$ from the strange susceptibility

bands (red and blue) indicate the widths of the transition lines
the widths remain in this order approximately the same
in leading order: no critical point (can be anything)

dashed line: freeze-out curve from experiments
Summary

- old result: QCD transition is an analytic cross-over
- long standing discrepancy in the literature
- overall scale $T_c$ was clarified (Wuppertal-Budapest)
- equation of state (EoS) was determined
- huge discrepancy between WB and hotQCD
- continuum limit of the phase diagram curvature $M_\pi = 135$ MeV
The nature of the QCD transition


finite size scaling study of the chiral condensate (susceptibility)

\[ \chi = \left( \frac{T}{V} \right) \frac{\partial^2 \log Z}{\partial m^2} \]

phase transition: finite V analyticity \( V \to \infty \) increasingly singular
(e.g. first order phase transition: height \( \propto V \), width \( \propto 1/V \))
for an analytic cross-over \( \chi \) does not grow with \( V \)

two steps (three volumes, four lattice spacings):
  a. fix \( V \) and determine \( \chi \) in the continuum limit: \( a=0.3,0.2,0.15,0.1 \) fm
  b. using the continuum extrapolated \( \chi_{\text{max}} \): finite size scaling
scaling regime is reached if $a^2$ scaling is observed
asymptotic scaling starts only for $N_t \gtrsim 8$ ($a \lesssim 0.15$ fm): two messages
a. $N_t=8,10$ extrapolation gives ’p’ on the $\approx 1\%$ level: good balance
b. stout-smeared improvement is designed to reduce this artefact
most other actions need even smaller ’a’ to reach scaling
Overlap improving multi-parameter reweighting

one wants to calculate the following path integral

\[ Z(\alpha) = \int [dU] \exp[-S_{bos}(\alpha, U)] \det M(U, \alpha) \]

\(\alpha\): parameter set (gauge coupling, mass, chemical potential)
for some parameters \(\alpha_0\) importance sampling can be done

\[ Z(\alpha) = \int [dU] \exp[-S_{bos}(\alpha_0, U)] \det M(U, \alpha_0) \]
\[ \{ \exp[-S_{bos}(\alpha, U) + S_{bos}(\alpha_0, U)] \det M(U, \alpha)/ \det M(U, \alpha_0) \} \]

first line: measure; curly bracket: observable (will be measured)
e.g. transition configurations are mapped to transition ones

rewighting factor (ratio of the determinants) can be expressed by the
eigenvalues of the (reduced) fermion matrix: closed formula for any \(\mu\)
Glasgow method $\Rightarrow$ multiparameter reweighting
single parameter ($\mu$) $\Rightarrow$ two parameters ($\mu$ and $\beta$)
purely hadronic $\Rightarrow$ transition configurations
map transition configurations to transition ones
All path approach

goal: determine the equation of state for several pion masses
reduce the uncertainty related to the choice of $\beta^0$
give the uncertainty related to the integration path

conventional path: A, though B, C or any other paths are possible
generalize: take all paths into account (use derivatives of $p$)
two-dimensional spline function gives $p$ for any $(\beta, R=m_s/m_{ud})$
technically: solution of a large system of linear equations
Finite chemical potential: the sign problem

at $\mu=0$ the fermion matrix is $\gamma_5$ hermitian: $M^\dagger = \gamma_5 M \gamma_5$
easy to check $\Rightarrow$ eigenvalues: either real or conjugate pairs

$\det(M)$ is real, which is not true any more for non-vanishing $\mu$

importance sampling (algorithms) for complex $\det(M)$ does not work

$P(U \rightarrow U') = \min[1, \exp(-\Delta S_g) \det(M[U'])/\det(M[U])]

sign problem $\Rightarrow$ until 2001: "lattice QCD can not say anything for $\mu>0$

Fodor-Katz: multiparameter reweighting (hep-lat/0104001, PLB)
Bielefeld-Swansee: $\det(M)$ Taylor expanded (hep-lat/0204010, PRD)
de Forcrand-Philipsen: imaginary $\mu$ (hep-lat/0205016, Nucl.Phys.B)
D’Elia-Lombardo: imaginary $\mu$ (hep-lat/0209146, PRD)

the three methods look different, they are essentially the same
**Equivalence of the methods (formal/numerical)**

(recent lattice review at $\mu=0$ and $\mu>0$: Fodor-Katz 0908.3341)

$\det(M)$ can be given by the eigenvalues of $M'$ (transformed) at $\mu=0$

$$\det M(\mu) = e^{-3V\mu} \prod_{i=1}^{6L^3} (e^{Lt\mu} - \lambda_i)$$

observable at $\mu>0$ or $\mu_I$ is given by the observable and $\lambda_i$ at $\mu=0$

$$Pl(\beta, \mu) = \langle Pl \exp[\Delta\beta Pl] e^{-3V\mu} \prod_{i=1}^{6L^3} (e^{Lt\mu} - \lambda_i) \rangle$$

$\det(M)$ or $Pl(\beta,\mu)$ can be trivially Taylor expanded (Bielefeld-Swansee) termination of the series & stochastic determination of the coefficients $\implies$ do not expect this method to work for as large $\mu$ as the full one

$\det(M)>0$ for imaginary $\mu$: impartment sampling still works
determine the phase line $T_c(\mu_I)$ (e.g. use a quadratic/quartic fit)
plug real $\mu$ into the same quadratic/quartic function: $c_2\mu^2 + c_4\mu^4$
formally: numerical determination of the $(\mu^2,\mu^4)$ Taylor coefficients

---

Z. Fodor  
QCD thermodynamics from the lattice: Continuum limit with...