Work and kinetic energy

1d motion: $v_x^2 - v_{x0}^2 = 2a_x(x - x_0) \quad | \times m/2$

$$\frac{mv_x^2}{2} - \frac{mv_{x0}^2}{2} = F_x(x - x_0) = F_x\,\Delta x$$

3d motion: $|\vec{v}|^2 = v^2 = v_x^2 + v_y^2 + v_z^2$ $\quad \Delta\vec{r} = (\Delta x, \Delta y, \Delta z)$

$$\frac{mv^2}{2} - \frac{mv_0^2}{2} = F_x\,\Delta x + F_y\,\Delta y + F_z\,\Delta z = \vec{F}\cdot\Delta\vec{r}$$

$$K - K_0 = W$$

Change in the *kinetic energy* equals to *work* done by the external force

Scalar (dot) product:

$$K \equiv \frac{mv^2}{2}$$

$$W = \vec{F}\cdot\Delta\vec{r}$$

$$\vec{a}\cdot\vec{b} \equiv a_x b_x + a_y b_y + a_z b_z$$

$$\vec{a}\cdot\vec{b} \equiv ab\cos(\theta_{ab})$$

Work done by constant force

- In physics, the *work* done on an object is the *scalar product* of the *force* exerted on that object multiplied by the *displacement* of that object.

- If the force exerted on an object is zero, or the displacement of the object is zero, no work is done.
  - ex: pushing down on a book sitting on a table does no work on the book, since the displacement is zero.
    - ex. A box is pushed across a level floor. The work done by gravity is equal to zero, because the displacement is at a right angle to the force.

- Work can be *positive* or *negative*, since $\cos\theta$ can be positive or negative.

- *Positive* work done on an object *increases* its energy, *negative* work done on an object *reduces* its energy.

- *Energy* is a scalar quantity associated with a system of objects, often defined as the ability to do *work*.
- The SI unit of energy is the Joule (J) where $J = N\ m = kg\ m^2/s^2$.

Work done by gravity

•The force of gravity on an object is *constant*, so we can use the equation $W_g = \mathbf{F} \cdot \mathbf{s} = Fs \cos\theta$ to find the work done by gravity (here θ is the angle between **F** and **s**).

•For a rising object, **s** is directed *upward*, and **F** is directed *downward*. This means that θ is 180°, so $\cos\theta = -1$.
In this case, $W_g = (F)(s)(-1) = -mgs$, where s is the *vertical* distance traveled by the object. Since W_g is *negative*, gravity is doing negative work on an object, decreasing its energy.

•When the object is *falling* back down, both **s** and **F** are directed downward so $\theta = 0°$ and $\cos\theta = +1$.
Then $W_g = (F)(s)(+1) = mgs$, where s is the vertical distance traveled by the object. Since W_g is positive, so gravity does an amount mgs of positive work on the object.

Work done by lifting

•Suppose an object is lifted up by applying a constant force $\mathbf{F}_a$ directed upwards, just sufficient to compensate for the force of gravity (that is, the acceleration of the object is always approximately zero).

•The work by the applied force, $\mathbf{F}_a$, through a vertical distance s is $W_a = F_a s$ (here **s** and **F** are in the same direction so $\cos\theta = +1$).

•The total work done on the object moving upward is the work done by the applied force plus the work done by gravity, $W_{net} = W_a + W_g = F_a s + (-mgs)$.

•This implies that there is no net work being one on the object; $W_a + W_g = 0$ so $W_a = -W_g$, the applied force transfers the exact same amount of energy to the object that the gravitational force removes from the object.

ex. How much work must you do to lift a 10 kg box to a height of 1.0 m, if the box begins and ends at rest (has $a \sim 0$ everywhere)?
$W_{app} = mgs = (10\text{ kg})(9.8\text{ ms}^{-2})(1\text{ m}) = 98\text{ J}$, so you must do +98 J of work to lift the box. Note that gravity is doing negative work, so the net work is zero.

Work done by a variable force

- 1D: For a general, non-constant force, we define the work done by the force, between initial position x_1 and final position x_2 as:

$$W = \int_{x_1}^{x_2} F(x)dx$$

- In 3D this equation can be generalized to:

$$W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}(\vec{r}) \cdot d\vec{r} = \int_{x_1}^{x_2} F_x dx + \int_{y_1}^{y_2} F_y dy + \int_{z_1}^{z_2} F_z dz$$

where the integral is calculated over the path the particle takes, F_x, F_y, and F_z are the components of force along the x, y, and z axes, and $\mathbf{r}_1 = (x_1, y_1, z_1)$ and $\mathbf{r}_2 = (x_2, y_2, z_2)$ are the initial and final positions.

Work done by a spring

- The force exerted by a spring is $F_x = -kx$, where k is the *spring constant*, and x is the displacement from the equilibrium position of the spring.

- The work done by the spring on an object as the object is moved from a displacement x_1 to a displacement x_2 is:

$$W_s = \int_{x_1}^{x_2} (-kx)\,dx = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2$$

- In the special case that x_1 is the equilibrium position of the spring, $x_1=0$ so $W_s = -\frac{1}{2}kx^2$.

- The work done by the spring is *positive* if the object ends up *closer* to the equilibrium position $(x = 0)$ and *negative* if the object ends up *further* away.