Collisions

- In *perfectly inelastic collisions*, the final velocities of the objects are identical.

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = (m_1 + m_2)\vec{v}_f \qquad \text{If } \mathbf{v}_{2i} = 0: \qquad \vec{v}_f = \frac{m_1}{m_1 + m_2}\vec{v}_{1i}$$

- If the collision is *elastic*, the total kinetic energy is conserved as well as momentum:

$$\tfrac{1}{2} m_1 v_{1i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2 \qquad m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

1d Elastic collisions

$$\text{If } \mathbf{v}_{2i} = 0: \qquad v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \qquad v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}$$

- If both objects are moving, the final velocities for an elastic collision are given by:

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i} \qquad v_{2f} = \frac{m_2 - m_1}{m_1 + m_2} v_{2i} + \frac{2m_1}{m_1 + m_2} v_{1i}$$

Special cases of 1d elastic collisions

If $m_1 = m_2$, then $v_{1f} = 0$ and $v_{2f} = v_{1i}$.

ex. When one billiard ball strikes another billiard ball at rest in a head-on collision (an elastic collision between objects with equal masses), the first billiard ball has final velocity zero, while the second billiard ball moves with the initial speed, and in the same direction, as the first billiard ball.

If $m_1 << m_2$, then $v_{1f} \sim -v_{1i}$ and $v_{2f} \sim 0$.

ex. When a ball is thrown against a wall, it bounces straight back with almost exactly the same speed with which is struck the wall. The wall doesn't move.

If $m_1 >> m_2$, then $v_{1f} \sim v_{1i}$ and $v_{2f} \sim 2v_{1i}$.

ex. For the above example, in the frame initially at rest with the ball, the initial speed of the ball is zero, and the wall approaches the ball with a speed v_{1i}. After the collision, the wall is will moving with a speed v_{1i}, but the ball is now moving in the same direction as the wall with a speed of $2v_{1i}$.

Collisions in two dimensions

- In *off-center* collision, the final velocities may not be *co-linear* (along the same axis).
- Consider such a collision, where object 2 is initially at *rest*.
- Defining the *x*-axis as the direction of $\mathbf{v}_{1i}$, and θ_1 and θ_2 to be the angles that $\mathbf{v}_{1f}$ and $\mathbf{v}_{2f}$ make with *x*-axis respectively, conservation of linear momentum gives:

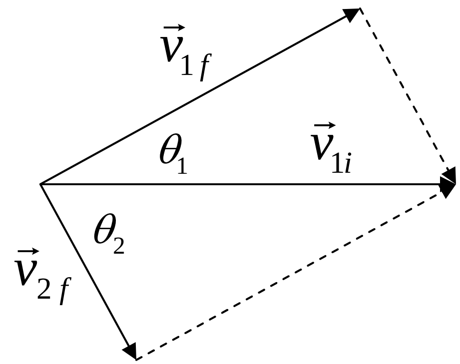
$$m_1 v_{1i} = m_1 v_{1f} \cos(\theta_1) + m_2 v_{2f} \cos(\theta_2)$$
$$0 = m_1 v_{1f} \sin(\theta_1) - m_2 v_{2f} \sin(\theta_2)$$

- If the collision is elastic, so kinetic energy is also conserved

$$\tfrac{1}{2} m_1 v_{1i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2$$

- In a special case $m_1 = m_2 = m$

$$\vec{v}_{1f} + \vec{v}_{2f} = \vec{v}_{1i}$$
$$v_{1f}^2 + v_{2f}^2 = v_{1f}^2$$



Center of Mass

$$\vec{F}_{net} = \frac{d\vec{P}}{dt} = \frac{d\left(\sum_{i}^{n} m_i \vec{v}_i\right)}{dt} = M\,\frac{d^2\vec{r}_{CM}}{dt^2} = M\,\vec{a}_{CM}$$

$$\vec{r}_{CM} = \frac{1}{M}\sum_{i=1}^{n} \vec{r}_i m_i$$

$$M = \sum_{i=1}^{n} m_i$$

•The *center of mass (CM)* of an object (or system) is the point in space representing the *average position* of all the mass of the system.

•The center of mass is important for two reasons: (i) the CM moves as though all of the systems mass were concentrated there, and (ii) one can consider all external forces as being applied to the CM for determining acceleration using Newton's Second Law.

> ex. If we throw a golf club up in the air, it is likely to tumble in flight, so neither the head nor handle will follow a parabolic trajectory. However, the center of mass of the golf club will describe a parabola.

•It is for these two reasons that we simply represent an object by a *point* in drawing free body diagrams.

Continuous mass distribution

$$\vec{r}_{CM} = \frac{1}{M}\sum_{i=1}^{n} \vec{r}_i m_i \;\Rightarrow\; \vec{r}_{CM} = \frac{1}{M}\int \vec{r}\, dm$$

•If the system has a continuous distribution of mass, we can no longer sum over discrete positions, so we need to change the sum into an integral.

•Again, defining M to be the total mass of the system, we define the *x*-, *y*-, and *z*-coordinates of the center of mass of the system as:

$$x_{CM} = \frac{1}{M}\int x\, dm,\;\; y_{CM} = \frac{1}{M}\int y\, dm,\;\; z_{CM} = \frac{1}{M}\int z\, dm$$

where dm is the mass element between $x+dx$, $y+dy$, or $z+dz$ as appropriate.

CM and uniform density

•The average density of a system is the mass of the system divided by the volume of the system ($\rho = M/V$).
•We define the local density of the system at each point as: $\rho = dM/dV$.
•If the density of the system is the same at every point in the system the system is said to be *uniform*.

$dm = \rho\, dV$	volume density
$dm = \sigma\, dA$	surface (area) density
$dm = \lambda\, dl$	linear density

•Because there is a linear relationship between volume and mass ($M = \rho V$) in *uniform systems*, the center of mass of a uniform system will be the same as the geometrical center of the object.

ex. The center of mass of a uniform sphere is the center of the sphere.

•For uniform systems, the center of mass will lie on point, line, or plane of symmetry (if such a point, line, or plane exists).

ex. A cone has a line of symmetry along the axis of the cone (rotating the cone about this line does not change the cone). Therefore, the center of mass of a uniform cone must lie along the axis.

Superposition and CM

•Once we know the center of mass of a system, we can treat the system as having all of its mass concentrated at this point.

•The net effect of the gravitational force acting on different parts of the same object can be treat as a single force of magnitude Mg acting at the center of mass.

•Because it is especially easy to find the CM of uniform systems with a high degree of symmetry, it can be useful to consider some systems with complex shapes as the sum of two (or more) different systems, each having a high degree of symmetry.

> ex. The CM of a snowman (approximated as three uniform spheres) can be computed by calculating the CM for each of the three spheres. Then, the CM for the snowman is simply the CM for three particles, having the masses of the three spheres, located at the CM for each sphere.

•For some problems, we will consider a system as being made of some other (more symmetric) system with positive uniform density, combined with another system with negative uniform density.

•*Note: we are not claiming that systems with negative uniform density exist, this is simply mathematical formalism that simplifies the problem.*

> ex. The compute the CM of a cube of uniform density ρ with a spherical hole located inside of it, we can compute the center of mass of a system consisting of a whole cube of density ρ and a sphere with density $-\rho$.

•This technique relies on the principle of *superposition*.