Center of Mass

$$\vec{F}_{net} = \frac{d\vec{P}}{dt} = \frac{d\left(\sum_{i}^{n} m_i \vec{v}_i\right)}{dt} = M\,\frac{d^2\vec{r}_{CM}}{dt^2} = M\,\vec{a}_{CM}$$

$$\vec{r}_{CM} = \frac{1}{M}\sum_{i=1}^{n} \vec{r}_i m_i$$

$$M = \sum_{i=1}^{n} m_i$$

- the CM moves as though all of the systems mass were concentrated there, and
- one can consider all external forces as being applied to the CM for determining acceleration using Newton’s Second Law.
- It is for these two reasons that we simply represent an object by a *point* in drawing free body diagrams.

Continuous mass distribution

$$\vec{r}_{CM} = \frac{1}{M}\sum_{i=1}^{n} \vec{r}_i m_i \;\Rightarrow\; \vec{r}_{CM} = \frac{1}{M}\int \vec{r}\, dm$$

$$x_{CM} = \frac{1}{M}\int x\, dm,\; y_{CM} = \frac{1}{M}\int y\, dm,\; z_{CM} = \frac{1}{M}\int z\, dm$$

CM and uniform density

•If the density of the system is the same at every point in the system the system is said to be *uniform*.

$dm = \rho\, dV$	volume density
$dm = \sigma\, dA$	surface (area) density
$dm = \lambda\, dl$	linear density

•For uniform systems, the center of mass will lie on point, line, or plane of symmetry (if such a point, line, or plane exists).

Superposition and CM

•Because it is especially easy to find the CM of uniform systems with a high degree of symmetry, it can be useful to consider some systems with complex shapes as the sum of two (or more) different systems, each having a high degree of symmetry.

ex. The CM of a snowman (approximated as three uniform spheres) can be computed by calculating the CM for each of the three spheres. Then, the CM for the snowman is simply the CM for three particles, having the masses of the three spheres, located at the CM for each sphere.

•For some problems, we will consider a system as being made of some other (more symmetric) system with positive uniform density, combined with another system with negative uniform density.

•*Note: we are not claiming that systems with negative uniform density exist, this is simply mathematical formalism that simplifies the problem.*

ex. The compute the CM of a cube of uniform density ρ with a spherical hole located inside of it, we can compute the center of mass of a system consisting of a whole cube of density ρ and a sphere with density $-\rho$.

•This technique relies on the principle of *superposition*.

Newton's Second Law for a system of particles

•Recalling that the CM moves as though all the mass were concentrated there, and one can consider all external forces as being applied to the CM, Newton's second law becomes:

$$\vec{F}_{net} = M\,\vec{a}_{CM}$$

- M is the *total mass* of the system
- $\mathbf{a}_{cm}$ is the acceleration of the *center of mass* of the system.

•This equation can be broken up into *x*, *y*, and *z* components.

•One special case of this equation occurs when $\mathbf{F}_{net} = 0$, so the center of mass of the system remains at a constant velocity.

ex. The center of mass of a bobsled and its two drivers is moving along a flat part of a track at a constant velocity of 25 m s^{-1}. The two drivers jump out of the bobsled, changing the velocities of the drivers and the sled itself. However, the CM of the two drivers and sled continues moving at 25 m s^{-1}.

•We also know (from the previous slide) that $M\,\mathbf{v}_{CM} = \mathbf{P}$ = constant when $\Sigma\mathbf{F}_{ext} = 0$.

Rocket propulsion

• We can use conservation of linear momentum to solve complicated systems, for example, systems with *varying mass*.
• Consider a rocket at time t with mass M+ Δm, of which Δm is fuel.
• Over a time Δt, the rocket having initial velocity v ejects Δm mass of fuel having a velocity v_e *relative to the rocket*, speeding up to $v+\Delta v$ in the process.
• Conservation of momentum gives us:

$$(M+\Delta m)v = M(v+\Delta v)+\Delta m(v-v_e)$$

• We can simplify this expression to:

$$Mdv = v_e dm = -v_e dM$$

• Which we can re-write as:

$$\int_{v_1}^{v_2} dv = -v_e \int_{M_1}^{M_2} \frac{dM}{M}$$

$$v_2 - v_1 = v_e \ln \frac{M_1}{M_2}$$