Rotation

$$\omega = \frac{d\theta}{dt}$$

$$\alpha = \frac{d\omega}{dt}$$

Rotation $\leftrightarrow$ Translation

$$\theta \Leftrightarrow x$$
$$\omega \Leftrightarrow v$$
$$\alpha \Leftrightarrow a$$

$$\omega T = 2\pi$$

$$v = \omega r$$

$$a_c = r\omega^2$$

Rotational kinetic energy

- When an object is undergoing rigid rotation, each point in the object is moving with a linear speed $v = r\omega$, where r is the distance of the point from the axis of rotation.
- Therefore, each point in the object will have some *kinetic energy* associated with it.
- Recall, the kinetic energy for a mass m moving with speed v is $K = ½\, mv^2$.

- Since the linear speed depends on the distance from the axis of rotation, different parts of the object are moving at different linear speeds. Therefore, we need to express the kinetic energy of the object in terms of the angular speed, which is the same for every point in the rigidly rotating object.
- As a start, consider an object made up of N particles (masses $m_1,\ldots,m_N$) at distances $r_1,\ldots,r_N$ from the axis of rotation. If the object is rotating with an angular velocity w, we know that $v_i = r_i\omega$ so

$$K_{rot} = \sum_{i=1}^{N} \tfrac{1}{2} m_i v_i^2 = \tfrac{1}{2}\left(\sum_{i=1}^{N} m_i r_i^2 \right)\omega^2$$

Inertia

$$K_{rot} = \sum_{i=1}^{N} \tfrac{1}{2} m_i v_i^2 = \tfrac{1}{2}\left(\sum_{i=1}^{N} m_i r_i^2\right)\omega^2$$

$$K_{rot} = \tfrac{1}{2} I\omega^2$$

$$I \equiv \sum_{i=1}^{N} m_i r_i^2$$

- I is called the *moment of inertia* (or *rotational inertia)*
(*I* takes the place of *m* and ω takes the place of *v*.)

$\theta \Leftrightarrow x$

$\omega \Leftrightarrow v$

$\alpha \Leftrightarrow a$

$I \Leftrightarrow m$

- In general, we define:

$$I = \int r^2 dm$$

where the integral is done over the the entire volume of the rotating object, and *r* is the distance of each mass element d*m* from the axis of rotation.

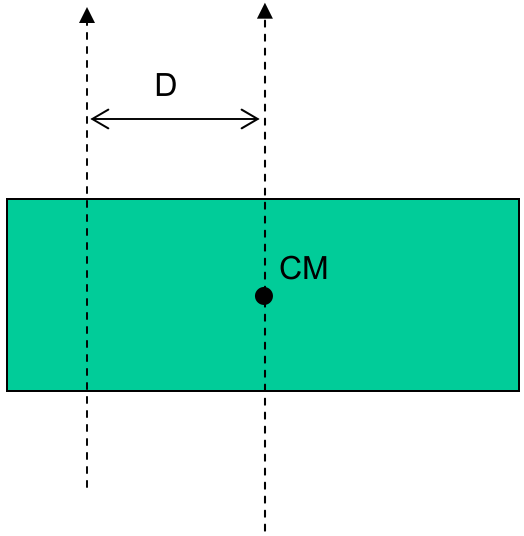
- In practice, solving this integral can be rather difficult. The moment of inertia for several regular objects is given in Table 10-2 of the textbook.
- Note that *I* depends on both the object being rotated *and on the axis of rotation*.
- If we know the moment of inertia for an object around an axis of rotation passing through the CM of the object, we can use the *parallel axis theorem* to find the moment of inertia about any other axis of rotation parallel to this first axis.

Parallel axis theorem

• The *parallel axis theorem* states that if I_{CM} is the moment of inertia of an object around an axis of rotation passing through the CM, then the moment of inertia around a different axis of rotation, *parallel* to this first axis, but *separated* by a distance D is:

$$I = I_{CM} + MD^2$$

where M is the mass of the object.



Torque

•*Torque* (τ) is the *rotational* analogue of *force*, in the sense that applying a net torque to a system produces an *angular acceleration*.

•A *force* acting on an object at some *distance* ($r=|\mathbf{r}|$) from the axis of rotation will produce a *torque* on the object according to the equation:

$$\tau = r\,F\,\sin\phi = Fd$$

where **r** is a vector from the rotation axis to the point of application of the force and ϕ is the angle between **F** and **r**.

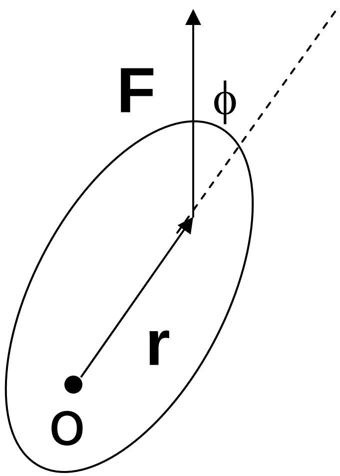
•Torque is *vector* quantity with *units* of N·m. Note that while formally N·m = J, we will never express torque in units of J because it is not an energy.

•A *positive* torque rotates the object *counterclockwise*, a *negative* torque rotates the object *clockwise*.

•If multiple forces are exerting torques on an object, we define the *net torque* (τ_{net}) as the sum of all torques acting on the object.

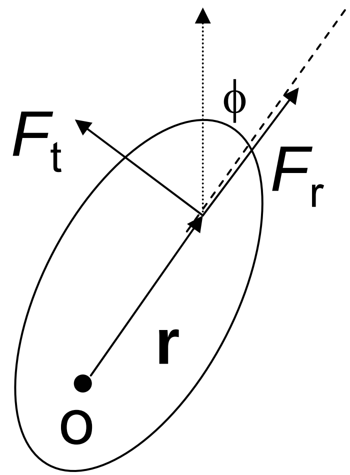
Properties of Torque

1. If F=0, τ=0 (you need a force acting on the object to produce a torque).
2. If s=0, τ =0 (a force acting at the axis of rotation will produce no torque).
3. If ϕ=0, τ =0 (a force acting in the same direction as **r** will produce no torque).
4. If F increases, τ increases (applying a larger force will produce a larger torque).
5. If s increases, τ increases (applying the force at a larger distance from the axis of rotation will produce a larger torque).



Axis of rotation

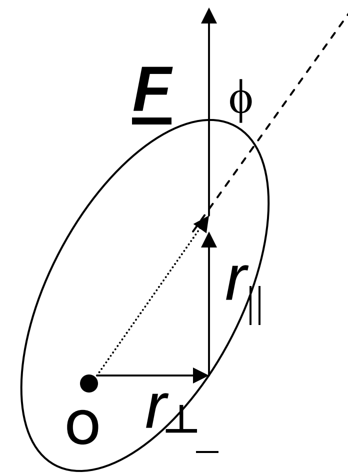
$\tau = Fr \sin\phi$



$F_t = F \sin\phi$

$\tau = F_t r$

F_t is the *tangential* component of F



$r_\perp = d = r (\sin\phi)$

$\tau = F r_\perp$

$r_\perp$ is the moment arm of **F**.

Newton's Second Law for Rotation

•Newton's Second Law for rotation states

$$\tau_{net} = I\,\alpha$$

• I is the moment of intertia of the object, and the angular acceleration of the object must be expressed in radian measure.

•Equivalently, a net torque acting on an object (around a certain axis of rotation) produces an angular acceleration (around this axis of rotation), where the angular acceleration is proportional to the net torque, and inversely proportional to the moment of inertia of the object (around this axis of rotation).

Proof

$F_t = ma_t$ (Newton's Second Law for linear motion).

$\tau = F_t r = m\, a_t\, r$ (where r is the distance from the center of rotation).

$a_t = \alpha r$ (conversion between angular and linear motion).

$\tau = m(\alpha r)r = (mr^2)\,\alpha = I\alpha$.

•*Important point*: τ, I, and α all depend on the *axis of rotation*. Make sure to choose an appropriate axis of rotation for the problem.

Work and Rotational Kinetic Energy

• Recall the *work-kinetic energy theorem* for linear variables states W = ΔK, the work done on an object equals the change in kinetic energy of the object (Remember: for the work-kinetic energy theorem, we consider work done by *both* internal and external forces).

• When a torque accelerates a rigid body about a fixed axis, this torque does *work* W. This changes the rotational kinetic energy of the body.

• We relate W to ΔK by the work-kinetic energy theorem for rotational variables

$$W = \Delta K = \tfrac{1}{2} I\omega_f^2 - \tfrac{1}{2} I\omega_i^2$$

• The work done by a *constant* torque is: $W = \tau\left(\theta_f - \theta_i\right)$

• More generally, the work done by a *variable* torque is:

$$W = \int_{\theta_1}^{\theta_2} \tau\, d\theta$$

• Since power is the rate of doing work, the *power* is:

$$P = \frac{dW}{dt} = \tau\omega$$

• Table 10-3 in the textbook summarizes pure translation vs pure rotation.