Rotation – vector notations

$$\vec{\omega} = \frac{d\vec{\theta}}{dt}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

$$\vec{\tau}_{net} = I\,\vec{\alpha}$$

$$I = \int r^2 dm$$

$$K_{rot} = \tfrac{1}{2} I\omega^2$$

$$\tau = F\, s\, \sin\phi$$

$$W = \int_{\theta_1}^{\theta_2} \tau d\theta$$

Rotation ↔ Translation

$\theta \Leftrightarrow x$

$\omega \Leftrightarrow v$

$\alpha \Leftrightarrow a$

$I \Leftrightarrow m$

$\tau \Leftrightarrow F$

$\vec{\omega}$

$\vec{\alpha}$

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\omega T = 2\pi$$

$$v = \omega r$$

$$a_c = r\omega^2$$

Cross (vector) product

- The dot (scalar) product produces a scalar from two vectors.
- The cross (vector) produce produces a vector from two vectors, where ***axb*** will be a *vector* ***c*** with *magnitude ab* sinϕ (with ϕ the angle between **a** and **b**) and pointing in a *direction perpendicular* to both **a** and **b**.
- To find the direction of **c**, imagine rotating vector **a** into vector **b** with your right hand. **c** = **a**x**b** will be in the direction of your extended thumb.

- The cross product is not commutative; **axb** = -**bxa**.

- The cross product is distributive: **a**x(**b**+**c**)=**axb**+**axc**.

- The derivative of the cross product is: $$\frac{d}{dt}\left(\vec{a}\times\vec{b}\right)=\frac{d\vec{a}}{dt}\times\vec{b}+\vec{a}\times\frac{d\vec{b}}{dt}$$

- From these definitions, if **a** is parallel (ϕ=0) or antiparallel (ϕ=180°) to **b**, then **axb**=0.

- This means that **axa** = 0
- If ***a*** is perpendicular to **b**, then |**axb**|= *ab*.

Cross (vector) product

- If **i**, **j**, and **k** are the orthogonal basis vectors for rectilinear coordinate, then:

i x **j** = **k**
j x **k** = **i**
k x **i** = **j**

- In component form, if **a** = (a_x, a_y, a_z) and **b** = (b_x, b_y, b_z),

axb = $(a_y b_z - a_z b_y,\ a_z b_x - b_z a_x,\ a_x b_y - b_x a_y)$.

- We can use the cross produce to more elegantly express the torque a force **F** exerts at a point displaced by **r** from a fixed point:

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

- This allows to do consider the torque relative to a fixed *point*, rather than a fixed *axis* of rotation. The torque can be in any direction, and the path followed by point does not need to be a circle.

Angular momentum

- Recall that the linear momentum of an object is $\mathbf{p}=m\mathbf{v}$.

- We define the *angular momentum* of a *particle* having linear momentum $\mathbf{p}$ about some point O as $\mathbf{L}= \mathbf{r}\times\mathbf{p} = m(\mathbf{r}\times\mathbf{v})$, where $\mathbf{r}$ is the displacement of the particle from point O.

- Note that the particle does *not* need to *rotate* about point O in order to have an angular momentum relative to O.

- Angular momentum is a *vector*, having *units* of kg m^2/s.

- The magnitude of the angular momentum is then $L = rmv\sin\phi$, where ϕ is the angle between $\mathbf{r}$ and $\mathbf{v}$.

- Equivalently, $L = rp_{\text{perp}}= mrv_{\text{perp}}$, where p_{perp} (v_{perp}) is the component of momentum (velocity) *perpendicular* to $\mathbf{r}$.

- The angular momentum of an object is always defined relative to a point O (which determines the “tail” of $\mathbf{r}$ vector).

Newton’s Second Law and angular momentum

- Recall that Newton’s Second Law for linear motion can be expressed as $\mathbf{F}_{net}=d\mathbf{p}/dt$.
- In a similar fashion, we can write *Newton’s Second Law* for rotational motion as:

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$$

so the time rate of change of angular momentum is equal to the *net* torque acting on the object.

Angular momentum for a system of particles

- The total angular momentum $\mathbf{L}_{tot}$ of a *system* of *particles* having angular momenta given by $\mathbf{L}_1,\ldots,\mathbf{L}_n$ is $\mathbf{L}_{tot}= \mathbf{L}_1+\ldots+\mathbf{L}_n$.
- Newton’s Second Law (angular form) for a system of particles states: $\vec{\tau}_{ext} = \dfrac{d\vec{L}_{tot}}{dt}$
- Where $\boldsymbol{\tau}_{ext}$ is the net external torque on the system (as a vector sum of all external torques), and $\mathbf{L}_{tot}$ is the total angular momentum of the system

Angular momentum

•Recall that the linear momentum of an object is **p** = *m***v**.

•We define the *angular momentum* of a *particle* having linear momentum **p** about some point O as **L**=**r**x**p** = m(**r**x**v**), where **r** is the displacement of the particle from point O.

•Note that the particle does *not* need to *rotate* about point O in order to have an angular momentum relative to O.

•Angular momentum is a *vector*, having *units* of kg m^2/s.

•The magnitude of the angular momentum is then $L = rmv \sin\phi$, where ϕ is the angle between **r** and **v**.

•Equivalently, $L = rp_{perp} = mrv_{perp}$, where p_{perp} (v_{perp}) is the component of momentum (velocity) *perpendicular* to **r**.

•The angular momentum of an object is always defined relative to a point O (which determines the "tail" of **r** vector).

Newton’s Second Law and angular momentum

- Recall that Newton’s Second Law for linear motion can be expressed as $\mathbf{F}_{net} = d\mathbf{p}/dt$.
- In a similar fashion, we can write *Newton’s Second Law* for rotational motion as:

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$$

so the time rate of change of angular momentum is equal to the *net* torque acting on the object.

Angular momentum for a system of particles

- The total angular momentum $\mathbf{L}_{tot}$ of a system of particles having angular momenta given by $\mathbf{L}_1, \ldots, \mathbf{L}_n$ is $\mathbf{L}_{tot} = \mathbf{L}_1 + \ldots + \mathbf{L}_n$.
- Newton’s Second Law (angular form) for a system of particles states: $\vec{\tau}_{ext} = \frac{d\vec{L}_{tot}}{dt}$
- Where $\boldsymbol{\tau}_{ext}$ is the net external torque on the system (as a vector sum of all external torques), and $\mathbf{L}_{tot}$ is the total angular momentum of the system

Angular momentum of a rigid body

- Consider a collection of objects (including continuous objects) rotating around some axis as a rigid body with constant angular speed ω.

- The contribution to **L** from some mass element Δm_i a distance r_i from the axis of rotation is $l_i = (r_i)(\Delta m_i v_i)$. Note that since we have *defined* r_i as the distance from the axis of rotation, r_i is *perpendicular* to v_i, so $\phi = 90^o$ and $\sin\phi = 1$.

Then, using $v = \omega r$, $l_i = (\Delta m_i)(r_i^2)\omega$.

- The total angular momentum of the system is: $L = \sum_{i=1}^{n} l_i = \omega \sum_{i=1}^{n} r_i^2 \Delta m_i$

- Recognizing the moment of inertia as $I = \sum_{i=1}^{n} r_i^2 \Delta m_i$

- We find that $L = I\omega$, where L and I are both relative to the axis of rotation of the system.

Conservation of angular momentum

•Recall that Newton's second law for *angular* motion can be expressed as $\boldsymbol{\tau}_{\mathrm{net}} = d\mathbf{L}/dt$.

•The *conservation of angular momentum* states that if the *net external torque* acting on a system is *zero*, the total angular momentum **L** of the system is *constant*.

•This is equivalent to saying that $\mathbf{L}_i = \mathbf{L}_f$ at all times t_i and t_f.

•Because angular momentum is a *vector* quantity, we can make the stronger statement that if the *component* of the net external torque on a system along a certain axis is *zero*, then the *component* of angular momentum along that axis is *constant*.

•Since $L = I\omega$, we can also express the conservation of angular momentum as $I_i\omega_i = I_f\omega_f$ (note that the *moment of inertia I* of the system can change).

•Do not forget that a particle moving in a *straight* line will have an angular momentum of $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, where **r** is the displacement of the particle from the axis of rotation, and **p** is the linear momentum of the particle.

Examples of angular momentum conservation

Figure skater spinning

One common example illustrating the conservation of angular momentum is a figure skating spinning. With her arms stretched out, a figure skater has moment of inertia I_i, and spinning at ω_i, has angular momentum $I_i\omega_i$. By drawing her arms closer to her body, she will decrease her moment to some value $I_f<I_i$. However, because her angular momentum cannot change, this means that her angular velocity ω must increase, so she starts spinning faster.

Neutron stars

When the nuclear reactions inside a star no longer provide enough outwards pressure, the gravitational forces in a star will cause it to collapse. The radius of the star changes from ~690 *thousand* kilometers to only a few kilometers (for a neutron star). However, the angular momentum of the star remains constant during this collapse. Therefore, since the moment of inertia is decreasing by many orders of magnitude (varying like R^2), the angular velocity will increase by many orders of magnitude. The angular speed of neutron stars can be almost 1000 revolutions *per second*, compared to one revolution per month for the Sun.

Riding a bicycle

Balancing on a bicycle at rest is very hard. You have to readjust your position to make sure that your center of mass is directly over a support point of the bike. However, balancing on a moving bicycle is very easy. When a bicycle is moving, the wheels have an angular momentum. Tipping the bicycle would change the angular momentum (by changing the *direction* of the angular velocity), requiring a large external torque. Since the magnitude of torque depends on the length of the lever arm, as long as you are anywhere near balancing (above a support point of the bicycle so the lever arm is very small), the small torque produced by mgr will not be sufficient to noticeably change the angular momentum of the wheels (on short time scales at least), therefore the bicycle will remain upright.