Gravity

- The most general expression for the attractive *gravitational force* between two masses is:

$$\boxed{F = G\frac{m_1 m_2}{r^2}}$$

where $G = 6.67\text{x}10^{-11}\ \text{m}^3\text{kg}^{-1}\text{s}^{-2}$ is called the gravitational constant, and r is the separation between the two masses.

- Near the Earth's surface, this can be approximated as $F = -mg$, where $g = 9.8\text{ms}^{-2}$.

- *Note 1*: A uniform shell of matter attracts a particle that is outside the shell as if all the shells mass were concentrated at its center.

- *Note 2*: A uniform shell of matter exerts no net gravitational force on a particle located inside of it.

Kepler's Laws for planetary motion

- *Kepler's Laws* for planetary motion follow from the universal expression for the force of gravity ($F_g = GMm/r^2$) and the *conservation of angular momentum*.

- They are derived assuming the Sun has mass *M*, which is much larger than the mass *m* of the orbiting planet (so that the CM of the system is *inside* the Sun).

- *Kepler's 1st Law*: All planets move in *elliptical* orbits, with the Sun at one *focus*.

- *Kepler's 2nd Law*: A line that connects the Sun to a planet sweeps out *equal areas in equal times* as the planet orbits the Sun.

- This is a consequence of the *conservation of angular momentum*.

- Recall that in angular coordinates, $\theta = s/r$. Then, the approximate area swept as the planet rotates through an angle $\Delta\theta$ is approximately $\Delta A \sim \frac{1}{2} rs = \frac{1}{2} r^2 \Delta\theta$, so dividing both sides by Δt, and approximating $\Delta y/\Delta t$ as dy/dt we find:

$$\boxed{\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \omega}$$

- The angular momentum of a planet around the sun is $L = mr^2\omega$.
- This leads to: $$\boxed{\frac{dA}{dt} = \frac{L}{2m}}$$
- Since L is constant, we have shown Kepler's 2nd law.
- *Kepler's 3rd law*: The *square* of the *period* of any planet is proportional to the *cube* of the *semimajor axis* of its orbit.
- Consider only circular orbits (so the semimajor axis is just the radius).
- Then $GMm/r^2 = m\omega^2 r$ (uniform circular motion).
- However, $\omega = 2\pi/T$, so we find: $$\boxed{T^2 = \left(\frac{4\pi^2}{GM}\right) r^3}$$

Gravitational potential energy

- Gravity is a conservative force, so we can define a gravitational potential energy as:

$$U_g = -\int_{r_1}^{r_2} G\frac{m_1 m_2}{r^2}dr = G\frac{m_1 m_2}{r_2} - G\frac{m_1 m_2}{r_1}$$

- If we consider $U_g = 0$ when the objects are infinitely far apart ($r_2 \to \infty$), the gravitational potential energy can be expressed as $U_g = -Gm_1m_2/r$.

- This leads to the definition of *escape velocity*—the minimum speed an object must have to escape the gravitational attraction of a planet.

- Because mechanical energy is conserved, the initial kinetic energy of the object must equal the change in potential energy of the object ($1/2\ m_1v^2 = Gm_1m_2/r^2$) so

$$v = \sqrt{\frac{2Gm_2}{R}}$$

where m_2 is the mass of the planet, and R is the radius of the planet.

Satellites and geosynchronous orbits

- Recall that an object in uniform circular motion is undergoing a centripetal acceleration with magnitude $a_c = v^2/r$, directed towards the center of rotation, and that the period for this motion is equal to $T = 2\pi r/v$.

- Consider a satellite of mass m in geosynchronous orbit, at a distance r from the center of the Earth, moving with constant speed v.

- The gravitational force acting on the satellite is: $$F_g = G\frac{mM}{r^2}$$

- This provides the centripetal acceleration, so: $$F_g = ma_c = 4\pi^2 m\frac{r}{T^2}$$

- We can solve these equations for r to obtain: $$r^3 = G\frac{MT^2}{4\pi^2}$$

- Giving $r = 42\,300$ km (measured from the center of the Earth).