Motion of a spring-mass system

- Recall *Hooke's Law*, which states that the force exerted by a *spring* is $F= -kx$.

- F is called a *restoring force*, since it always directed towards the equilibrium point of the spring, x=0 (it acts to *restore* the equilibrium).

- If a spring force is the *only* (net) force acting on an object of mass m, $F=ma= -kx$.

- This means that $a_x=-(k/m)x$, so the acceleration of the block is proportional to the displacement, and in the opposite direction to this displacement.

- Systems exhibiting this type of motion are said to be undergoing *simple harmonic motion*.

- Note that when $x=0$, $a=0$ so the velocity is at a maximum.

- If the mass is displaced to $x=A$ and released from rest, the maximum acceleration is equal to $|\underline{\boldsymbol{a}}|=|kA/m|$

Simple Harmonic Motion

- For simple harmonic motion, we had $a=-(k/m)x$, which can also be expressed as:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x$$

where we have made the definition $\omega^2=k/m$.

- This equation can be solved by:

$$x(t) = A\cos(\omega t + \phi)$$

where A (the *amplitude* of the motion) and ϕ (the *phase constant*) are constants, typically determined from the initial conditions given in the problem.

NB We have a second order differential equation, so there are two constants.

- The amplitude is the maximum position of the object in the positive or negative x directions (that is, $-A \leq x \leq A$)

Period and frequency

- The constant ω is called the *angular frequency* of the simple harmonic motion, and has units of rad/s. Note that $\omega = (k/m)^{1/2}$, so this is fixed for a given system, and does not depend on the initial conditions for the simple harmonic motion.

- The phase constant ϕ gives the initial position of the object on the sinusoidal curve.

- The period T of simple harmonic motion is the time it takes for the object to go through one complete cycle of motion, or $[\omega(t+T)+\phi] - (\omega t+\phi)=2\pi$, which can be solved to give:

$$T = \frac{2\pi}{\omega}$$

- We define the frequency f of the motion as $f=1/T=\omega/2\pi$. This gives the number of oscillations the object undergoes in a unit time interval.

- f can also be expressed as: $\omega = 2\pi f = (1/2\pi)(k/m)^{1/2}$

- Since $v(t)=dx(t)/dt$, we find that for simple harmonic motion:

$$v(t) = -\omega A \sin(\omega t + \phi)$$

- The velocity varies between $-\omega A$ and $+\omega A$ (so ωA is the *velocity amplitude*).

- The velocity of the object also describes a *sinusoidal* wave, but is shifted from x(t) by *one quarter period*.

- This means that when $x(t)$ is at a *maximum*, $v(t)=0$, but when $x(t)=0$, $v(t)$ is at a *maximum*.

- Since $a(t)=d^2x(t)/dt^2$, the *acceleration* of an object undergoing SHM is:

$$a(t) = -\omega^2 A \cos(\omega t + \phi)$$

- The acceleration also describes a *sinusoidal* wave, but is shifted from $x(t)$ by *one half period*.

- The maximum velocity for an object in SHM is $v_{\max}=\omega_A$, the maximum acceleration for on object in SHM is $a_{\max}=\omega^2 A$.

Energy in Simple Harmonic Motion

- Recall that there is potential energy stored in a stretched (or compressed) spring equal to $U(x)=\frac{1}{2} kx^2$.

- Since $x(t)=A\cos(\omega t+\phi)$ for an object undergoing SHM, the spring potential energy will also be a function of time, namely $U(t)=\frac{1}{2} kA^2\cos^2(\omega t+\phi)$.

- The mass will also have kinetic energy given by $K(v)=\frac{1}{2} mv^2$.

- Using our expression for $v(t)$ we find $K(t)=\frac{1}{2} kA^2\sin^2(\omega t+\phi)$, where we have used the fact that $k=m\omega^2$.

- Then, the *total mechanical energy* in the system is $E=K+U=\frac{1}{2}kA^2$ (using the fact that $\sin^2(\theta)+\cos^2(\theta)=1$).

- The distribution of this mechanical energy will change between potential energy (largest at the $x=\pm A$) and kinetic energy (largest at $x=0$).

- As a function of position, the velocity is given by: $v=\pm\omega(A^2-x^2)^{1/2}$.

SHM and Uniform Circular Motion

- There is a strong relationship between simple harmonic motion and uniform circular motion.

- Namely, SHM with amplitude A and angular frequency ω is just the projection onto the x axis of a point moving around a circle, centered at (0,0) with radius A, with a uniform speed given by $v=A\omega$.

- Note that the projection of this uniform circular motion onto the y axis would also yield simple harmonic motion (of the y-component of position of the object).

- Alternatively, *uniform circular motion* can be considered a *combination* of *two* simple harmonic motions *differing in phase* by 90^o (here, the x and y axes).

Pendulums

- Consider a mass *m* suspended from a light string of length L, which is connected to the ceiling at a pivot point P.

- If this mass is displaced from its equilibrium position, the force of gravity will exert a torque on *m* about P.

- This torque is equal to $\tau = -L(mg\sin\theta)$, where θ is the angle through which the mass is displaced from equilibrium.

- Newton's Second Law in angular form states that $\tau = I\alpha$, so we find that:

$$\alpha = -\frac{mgL\sin\theta}{I}$$

- Note that this does ***not*** describe simple harmonic motion, because the acceleration is ***not*** simply a negative number times the angular displacement.

- However, if θ is small, $\sin\theta \sim \theta$ so we can write:

$$\alpha = -\frac{mgL\theta}{I}$$

which does describe SHM.

- Comparing this expression with our equation defining SHM ($\alpha=-\omega^2\theta$ in this case), and using $\omega T=2\pi$, we find that:

- For this particular example, the moment of inertia I of the mass m about the pivot point P is simply $I=mL^2$.

- Therefore, the *period* of the motion is $T=2\pi(L/g)^{1/2}$.

- We can generalize this result to more complex *extended* objects.

- Consider an object of mass M free to rotate about a point P, having moment of inertia I (relative to P), and where the COM is a distance d from P.

- Then, if the object is rotated through an angle θ (about the axis passing through P) from its equilibrium position, the force of *gravity* will exert a torque $\tau= -Mgd\sin\theta$ on the system (about the axis passing through P).

- Then for small θ, $\alpha=-Mgh\theta/I$, so the object will undergo SHM with:

$$T = 2\pi\sqrt{\frac{I}{Mgd}}$$

Damped Simple Harmonic Motion

- Recall that in many physical systems, we can approximate the *drag force* acting on an object moving relative to a fluid as $F_d=-bv$, where v is the *relative velocity* of the object and fluid and b is the *damping constant*.

- Suppose the object is also acted on by a spring force $F_s= -kx$.

- Then, using Newton's Second Law $F_{net}=F_d+F_s=ma$ so: $-kx-bv=ma$

or $$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0$$

- The solution of this equation is: $x(t)=x_m e^{-bt/2m}\cos(\omega' t+\phi)$

where $$\omega'=\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}$$

is the *angular frequency* of the oscillation.

Damped SHM (ii)

- Note that adding a damping term to the motion of an object changes the angular frequency.

- However, if $b^2/4m^2 << k/m$ (so $b^2 << 4km$), then $\omega \sim \omega'$, and the angular frequency is almost that same as would be measured for an *undamped* oscillator.

- If $b^2/4m^2 > k/m$, the angular frequency is *not real* and the motion is said to be *overdamped*.

- If the damping is small ($b^2 << 4km$) then we can replace x_m by $x_m e^{-bt/2m}$ in the expression for the total mechanical energy of the system and find that:

$$E(t) \approx \tfrac{1}{2} k x_m^2 e^{-bt/m}$$

for systems undergoing damped simple harmonic motion.

- Therefore, in systems undergoing damped SHM, *both* the *amplitude* of the oscillations and the *mechanical energy* in the system decrease exponentially with time.

Forced Oscillations and Resonance

•Consider a system having some *natural* angular frequency ω (the intrinsic, undamped frequency of the system), with some *damping* $F_d = -bv$ which is acted on by a time-varying *driving* force given by $F_{app} = -F_m \cos(\omega_p t)$.

•Then the equation of motion of the system will be:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + m\omega^2 x + F_m \cos(\omega_p t) = 0$$

•The solution to this equation is complicated, but we can make a couple of statements about the general properties of forced, damped harmonic motion

1. The amplitude of the oscillations is *greatest* when $\omega \sim \omega_p$ (the driving force acts at the natural frequency of the system), and
2. This maximum amplitude for the oscillations varies *inversely* with *b*.

•The condition $\omega \sim \omega_p$ is called *resonance*, and will lead to very large oscillations.

ex. Pushing a swing at the natural pendulum frequency will lead to large amplitudes of the oscillation.