Wave equation

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$$

$$\boxed{y(x,t) = y_m \sin(kx - \omega t + \phi)}$$

$$\boxed{v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f}$$

$$\boxed{k\lambda = 2\pi}$$

Superposition of Waves

- Consider two waves traveling down the same length of string, defined by $y_1(x,t)$ and $y_2(x,t)$ respectively.

- The *principle of superposition* states that the resultant displacement of the string is described simply by $y(x,t) = y_1(x,t)+y_2(x,t)$.

- This is equivalent to the statement that the waves defined by y_1 and y_2 are *independent*; they do not alter the propagation of one another and the resultant displacement is the *algebraic* sum of the two individual displacements.

- The fact that the net displacement of a section of string is the sum of the two (or more) individual displacements from each waves leads to the concept of *interference*.

- Suppose $y_1(x,t) = y_m \sin(kx-\omega t)$ and $y_2(x,t) = y_m \sin(kx-\omega t+\phi)$, then the net displacement at each point is given by:

or

$$\boxed{\begin{aligned} y'(x,t) &= A\sin\left(kx-\omega t\right)+A\sin\left(kx-\omega t+\phi\right)= \\ &=\left[2A\cos\left(\tfrac{1}{2}\phi\right)\right]\sin\left(kx-\omega t+\tfrac{1}{2}\phi\right)\end{aligned}}$$

- This expression shows that if two sinusoidal waves with the *same wavelength* and *amplitude* travel in the *same* direction, they interfere to produce a sinusoidal wave traveling in the *same* direction.

- This resultant wave will have an amplitude $y_m' = 2A\cos\frac{1}{2}\phi$, and travel with the same frequency.

- If $\phi = 0$, the two waves undergo *fully constructive interference*, and the magnitude of the resultant wave is *maximal*.

- If $\phi = \pi$, the two waves undergo *fully destructive interference*, and the amplitude of the resultant wave is equal to *zero*.

- For intermediate values of ϕ, we say that the two waves undergo *intermediate interference*, and the amplitude of the resultant wave will be $0 < A' < 2A$.

- Recall, this derivation assumed that the two waves had the *same* amplitude and wavelength, and differed only by a *phase* ϕ.

Interference

- We can calculate the *interference* between sound waves emitted from different point sources using the *superposition principle*.

- Consider two *identical* point sources S_1 and S_2 located at s_1 and s_2, which emit sound waves with the same initial phase (ϕ=0 for both sources).

- The sound waves emitted from both S_1 and S_2 interfere at a point *P*, located a distance L_1 from S_1 and L_2 from S_2.

- At point *P* the sound emitted from S_1 will have a different *phase* from the sound emitted by S_2. This *phase difference* depends on the difference in path length ($\Delta L = L_2 - L_1$) as:

$$\phi = 2\pi \frac{\Delta L}{\lambda}$$

- *Fully constructive interference* occurs when ϕ is an integer multiple of 2π (ϕ=0, 2π, 4π,...); *fully destructive interference* occurs when ϕ is an odd multiple of π ($\phi=\pi$, 3π, 5π,…).

- Equivalently, *fully constructive interference* occurs when $\Delta L/\lambda$=0,1,2,…, and *fully destructive interference* occurs when $\Delta L/\lambda$=1/2, 3/2, 5/2, …

Standing Waves

- If two waves having the *same* amplitude and wavelength are traveling down the same piece of string in *opposite* directions, the interference will produce *standing waves* (that are not *traveling* waves).

- If $y_1(x,t) = A_1\sin(kx-\omega t)$ and $y_2(x,t) = A_2\sin(kx+\omega t)$ the superposition principle gives the resultant displacement as:

$$\boxed{y'(x,t) = \left[2A\sin(kx)\right]\cos(\omega t)}$$

- Note that in this expression the phase does not depend on $kx - \omega t$, so this does not define a traveling wave.

- *Nodes*, for which the resultant displacement of the string is *zero*, are given by $kx = n\pi$ or $x = n\lambda/2$

- *Antinodes* (located halfway between adjacent nodes) are sections of the string which have the *largest displacement* amplitude, and are given by $x = (n+½)\lambda/2$.

Standing Waves and Resonance

- Consider a series of pulses in a string with boundaries (whether *free* or *fixed*) at both ends. The incident and reflected pulses travel back and forth along the string, *interfering* with each other.

- For certain special wavelengths, the interference between all the incident and reflected pulses will produce *standing waves*, where the maximum amplitude of oscillation of the string depends *only* on x position (equivalently, the positions of the *nodes* are *stationary*).

- These standing waves occur at *resonance*, and the *frequencies* giving rise to standing waves are called *resonant frequencies*.
- The resonant frequencies are found by recalling that *adjacent nodes* are separated by *one half wavelength* and that *nodes* and *antinodes* are separated by *one quarter wavelength*.

- Therefore, for a string of length L *fixed* at both ends (the the boundaries are antinodes), the wavelength at resonance must satisfy:

$$\lambda = \frac{2L}{n}, \quad n = 1, 2, 3, \ldots$$

so

$$\boxed{f = \frac{v}{\lambda} = n\frac{v}{2L}}$$

- Converting wavelength into frequency, we find that the resonant frequencies for a string of length L fixed at both ends are:

$$\boxed{f = \frac{v}{\lambda} = n\frac{v}{2L}}$$

- Here, the wave with the lowest frequency (n=1) is called the *fundamental mode* or *first harmonic*. In general the j^{th} *harmonic* is the mode with $n = j$.

- The collection of all possible standing waves is called the *harmonic series*.

- These expressions for the harmonic modes were determined for systems with fixed boundary conditions. The conditions for resonance will be different in systems with different boundary conditions (ex. one free end and one fixed end, or where both ends are free).

Standing Waves in Air Columns

- A tube with air in it will support standing sound waves at different resonant frequencies.

- At a *closed* end of the tube, there must be a *node* in the standing wave; at an *open* end of the tube, there must be an *antinode* in the standing wave.

- Consider a tube of length *L* *open* at *both* ends. For a standing wave, *both* ends of the tube must be an *antinode*. Since antinodes are separated by $\lambda/2$, we find that the first harmonic of this open tube is $\lambda = 2L$.

- Higher harmonics are given by $\lambda = 2L/n$, where $n = 1, 2, \ldots$

- For a tube of length *L* with one end *closed* and one end *open*, the harmonics are defined by $\lambda = 4L/n$, where $n = 1, 3, 5, \ldots$ ($2m+1$ for m an integer).

- In practice, more than one harmonic of a string or tube can be excited at the same time, giving more complicated waveforms.

Beats

- The term *beats* refers to periodic variations in sound intensity arising from the *interference* between two sound waves with *different* frequencies.

Consider two sound waves having the *same* intensity at some position defined by:
$s_1 = s_m \cos(\omega_1 t)$ and $s_2 = s_m \cos(\omega_2 t)$, with $\omega_1 > \omega_2$

- Using the *superposition principle* the net displacement is the sum of these two terms, $s_{net} = s_1 + s_2$.

- This is equivalent to: $s = 2s_m \cos\left[\tfrac{1}{2}(\omega_1 - \omega_2)t\right]\cos\left[\tfrac{1}{2}(\omega_1 + \omega_2)t\right]$

- Writing $\omega' = ½(\omega_1 - \omega_2)$ and $\omega = ½(\omega_1 + \omega_2)$ we find:

$$\boxed{s(t) = \left[2 s_m \cos(\omega' t)\right]\cos(\omega t)}$$

- The maximum amplitude will occur whenever $|\cos\omega' t| = 1$ so when $\omega' = 0$ or $\omega' = p$.

- Therefore, the *beat frequency* of the resultant wave is $f_{beat} = 2\omega' = \omega_1 - \omega_2$.