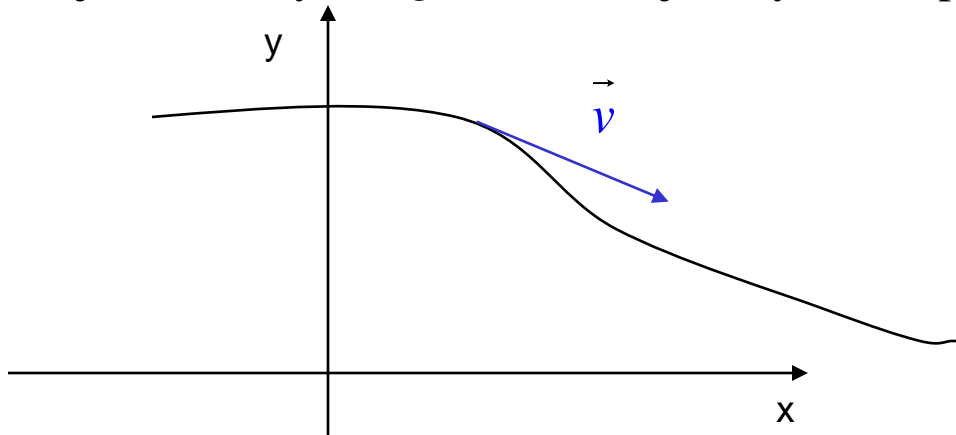


Motion in two and three dimensions

- The position of an object in three dimensional space is represented by a vector \mathbf{r} . We will refer to $\mathbf{r}(t)$ as the *trajectory* of the object.
$$\mathbf{r}(t) = r_x(t)\hat{\mathbf{i}} + r_y(t)\hat{\mathbf{j}} + r_z(t)\hat{\mathbf{k}}$$
- In usual Cartesian coordinates, we can decompose this vector as
- The *displacement* of this object between two positions $\mathbf{r}_1 = (x_1, y_1, z_1)$ and $\mathbf{r}_2 = (x_2, y_2, z_2)$ is a vector,
$$\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$
- We define the *average velocity* over a time interval Δt as $\mathbf{v}_{\text{avg}} = \Delta\mathbf{r}/\Delta t$
- The *instantaneous velocity* of an object with position $\mathbf{r}(t) = (x(t), y(t), z(t))$ is:

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}, \frac{dz(t)}{dt} \right)$$

- Note that this equation leads to the observation that the direction of the instantaneous velocity of an object is always *tangent* to its trajectory at that point



Acceleration

- The instantaneous acceleration of an object with a trajectory $\mathbf{r}(t)$, and velocity $\mathbf{v}(t)$ is:

$$\vec{a} = \frac{d\vec{v}(t)}{dt} = \frac{d^2\vec{r}(t)}{dt^2} = \left(\frac{d^2x(t)}{dt^2}, \frac{d^2y(t)}{dt^2}, \frac{d^2z(t)}{dt^2} \right)$$

- Note that since the acceleration is the derivative of velocity, the acceleration is *non-zero* if *either* the speed (magnitude of velocity) *or* the direction of \mathbf{v} of the object is changing in time.

ex. An object moving in a circle with constant speed is *accelerating*.

- Graphically, the direction of acceleration is *tangent* to the velocity curve at that point.

Solving 2d and 3d kinematics problems

- Since x , y , and z are orthogonal, the motion along one of these directions can be solved for independently of the motion along the other directions.
- When possible, it is easiest to decompose the motion of an object into x , y , and z coordinates.
- This leads to a system of two equations (2d motion) or three equations (3d motion) to solve.
- We will often parameterize the trajectory of an object as a function of *time* (t). It is important to realize that the x , y , and z coordinates of the object at any point along the trajectory are determined at the same *time*.

2d motion with constant acceleration

- Motion in 2d can be treated as independent 1d motions along the x and y directions *having the same time variable t* .
- If the 2D acceleration \mathbf{a} is constant, the x and y components of acceleration \mathbf{a}_x and \mathbf{a}_y are also constant. (Why?)
- Since the two 1D equations of motion along the x and y directions are independent, we can use our 1D constant acceleration equations to find:

$$r_x = x_0 + v_{x,i} t + \frac{1}{2} a_x t^2$$

$$r_y = y_0 + v_{y,i} t + \frac{1}{2} a_y t^2$$

or
$$\vec{r}(t) = \vec{r}_0 + \vec{v}t + \frac{1}{2} \vec{a}t^2 \quad \text{with } \mathbf{r}_0 = (x_0, y_0)$$

Projectile motion

- One particular important example of 2d motion is *projectile motion*.
- This type of motion occurs when an object is given some initial velocity \mathbf{v}_0 , and then accelerates only under the influence of gravity.
 - ex. A 3-point basketball shot, football punt, and home-run hit are all examples of projectile motion.
- We break up the two dimensional equation of motion, into two one-dimensional equations of motion, linked by the common *time* variable t .
 - ex. The 2d equation of motion for an object is $\mathbf{r}(t) = \mathbf{v}_0 t + \frac{1}{2} \mathbf{a} t^2$.
Using the convention that +y is up, this is equivalent to
 $x(t) = v_{0x} t$, $y(t) = v_{0y} t - g t^2 / 2$.
- Given the magnitude of the initial velocity v_0 , and the initial angle θ_0 between \mathbf{v}_0 and the +x direction, $v_{0x} = v_0 \cos\theta_0$, and $v_{0y} = v_0 \sin\theta_0$.
- Since the acceleration is *only* in the -y direction, $v_x = v_{0x}$; the **horizontal** component of velocity is **constant**.

Equations for projectile motion

- Assume an initial velocity v_0 at an angle of θ_0 to the $+x$ direction, with gravity acting in the $-y$ direction. The last equation assumes that $x_0=0$ and $y_0=0$.

$$v_x = v_0 \cos\theta_0$$

$$v_y = v_0 \sin\theta_0 - gt$$

$$x - x_0 = (v_0 \cos\theta_0)t$$

$$y - y_0 = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2$$

$$y = (\tan\theta_0)x - \frac{gx^2}{2(v_0 \cos\theta_0)^2}$$

- At the *maximum* height of the trajectory, $y'(t) = 0$, which implies that $v_y(t) = 0$, while $v_x(t) = v_0 \cos\theta_0$.

NB Since $y''(t) = -g < 0$ this is a *maximum*, not a *minimum*.

- Note that if $v_y(t) \neq 0$ for all physical values of t , then the maximum height occurs at the *beginning* or *end* of the trajectory.

Projectile motion range

- The *horizontal range* (or just *range*) of a projectile is the *horizontal* distance traveled by the object when it returns to its *initial height*.
- Note that this may be different that the *total* horizontal distance traveled.
Ex. If a ball is launched with the same initial velocity from a 100 m high pedestal, it will travel farther horizontally than it would if launched from the ground.
- The range of a projectile can be found by solving the equation $y-y_0=0$, and taking the non-zero solution for $x-x_0$.
NB. $y-y_0=0$ at $t=0$, which implies that $x-x_0=0$. This is not the range of the projectile, unless $v_0=0$.
- The range of a projectile is given by:
$$R = \frac{v_0^2}{g} \sin 2\theta_0$$
- The maximum range (for a certain v_0) occurs when $\theta_0=45^\circ$