Viscosity in strongly coupled gauge theories
Lessons from string theory

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A review of many people’s work
Basic setup

Relativistic field theories:

In $d = 3+1$ dimensions

Translation + rotation invariant

Have stable thermal equilibrium state

For simplicity $\mu = 0$, not near critical point

Small fluctuations on top of thermodynamic equilibrium state:

Slowly relaxing d.o.f.: $\epsilon = T^{00}$, $\pi^i = T^{0i} = \langle \epsilon + P \rangle v^i$, $\rho = J^0$

Obey classical equations — hydrodynamics

Does not depend on microscopic details of the theory
Hydrodynamic fluctuations

- Conservation laws: \( \partial_\mu T^{\mu\nu} = 0 \) \( \Rightarrow \)
  \[
  \begin{align*}
  \partial_t \epsilon &= -\nabla \cdot \pi \\
  \partial_t \pi^i &= -\nabla_j T^{ij}
  \end{align*}
  \]

- Constitutive relations:
  \[
  \begin{align*}
  T^{ij} &= \delta^{ij} \left[ \langle P \rangle + v_s^2 \delta \epsilon - \gamma_\zeta \nabla \pi \right] - \gamma_\eta \left( \nabla^i \pi^j + \nabla^j \pi^i - \frac{2}{3} \delta^{ij} \nabla \pi \right) + \ldots \\
  \gamma_\eta &\equiv \frac{\eta}{\langle \epsilon + P \rangle}, \quad \gamma_\zeta \equiv \frac{\zeta}{\langle \epsilon + P \rangle}, \quad v_s^2 = \partial P / \partial \epsilon
  \end{align*}
  \]

- Viscosities \( \eta, \zeta \) — input from microscopic physics

Two eigenmodes:

Shear mode: \( \pi_\perp (t, k) = e^{-\gamma_\eta k^2 t} \pi_\perp (0, k) \)

Sound mode: \( \pi_\parallel (t, k) = e^{-\frac{1}{2} \left( \gamma_\zeta + \frac{4}{3} \gamma_\eta \right) k^2 t} \times \)
\[
\times \left[ \pi_\parallel (0, k) \cos(kv_s t) - i kv_s \sin(kv_s t) \delta \epsilon (0, k) \right]
\]

Long-wavelength response is controlled by a small number of kinetic coefficients
Correlation functions in the hydrodynamic limit

Hydrodynamic modes ⇒ hydrodynamic singularities at small $\omega$, $k$.

Example: $S_{tx,tx}(\omega, k) = \frac{2\gamma \eta k^2}{\omega^2 + (\gamma \eta k^2)^2} (\epsilon + P) T$ relaxation of transverse momentum

Kubo formulas

$$\eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int dt e^{i\omega t} \int d^3 x \langle [T_{xy}(t, \mathbf{x}), T_{xy}(0)] \rangle$$

Connection to microscopic physics: Viscosities can be extracted from (small-frequency limits of) real-time correlation functions
Physical meaning of viscosity

Example: flow of gas

Internal friction — momentum exchange between layers of fluid

\[ T_{zx} = \eta \frac{\partial v}{\partial z} \]  
flux of \( x \)-momentum in \( z \)-direction

Particle number per unit surface per unit time: \((n v_{\text{th}})\)

Momentum per unit surface per unit time: \((n v_{\text{th}})(m v)\)

Flux up: \( n v_{\text{th}} m (v_0 - l_{\text{mfp}} \frac{\partial v}{\partial z}) \), flux down: \( n v_{\text{th}} m (v_0 + l_{\text{mfp}} \frac{\partial v}{\partial z}) \),

\[ T_{zx} = \text{flux}_{\text{up}} - \text{flux}_{\text{down}} \]

\[ \eta \sim n v_{\text{th}} m l_{\text{mfp}} \sim \frac{m v_{\text{th}}}{\sigma} \sim \frac{(m T)^{1/2}}{\sigma} \]

Ideal gas has infinite viscosity
How viscosity is measured

WEAK COUPLING

Prof. J. Maxwell, Mrs. Maxwell, (attic, 1860)

STRONG COUPLING

RHIC, many people (Brookhaven, 2000)

Strong coupling is more difficult
How viscosity is computed

- **Gases:** Kinetic theory
  \[ \eta \propto (mT)^{1/2} \frac{1}{\sigma} \]
  Density-independent, grows with \( T \)

- **Plasmas:** Boltzmann-Vlasov equation
  \[ \eta \propto (mT)^{1/2} \left( \frac{T^2}{e^4 \ln \Lambda} \right) \]
  “Coulomb log” \( \Lambda \sim \frac{l^4 T}{e^2} \sim \frac{T^{3/2}}{e^3 n^{1/2}} \) due to small-angle scatt.

- **Liquids:** Complicated: use molecular dynamics
  \[ \eta \propto \exp\left( \frac{E_a}{T} \right) \]
  Typical: decreases with \( T \)

- **Weakly coupled QFT:** Boltzmann-Vlasov equation, or resum Feynman diagrams
  \[ \eta \propto \frac{T^3}{g^4 \ln g^{-1}} \]
  (Log absent if no gauge fields)

- **Non-abelian gauge theories:** at \( T \gtrsim T_c \) need lattice
  Evaluate Euclidean correlators, invert to find real-time spectral functions...
  \[ \eta = ??? \]
Q: Are there any toy models where viscosity can be analytically computed at strong coupling?

A: \( \mathcal{N}=4 \) supersymmetric Yang-Mills theory, \( SU(N_c) \), \( \lambda = g^2 N_c \)

Why \( \mathcal{N}=4 \) SYM?
- Dual string description (AdS/CFT correspondence) — allows to compute correlation functions at strong coupling

What is \( \mathcal{N}=4 \) SYM?
- Gauge fields + 4 fermions + 6 scalars in adjoint of \( SU(N_c) \)
- Conformal theory, \( \lambda \) is a tunable parameter (does not run)
- Supersymmetric, but SUSY not essential at finite temperature
- \( \epsilon = 3P, v_s = \frac{1}{\sqrt{3}}, \zeta = 0 \), at any non-zero temperature
- \( \epsilon, P, \eta \) are finite in the limit \( \lambda \to \infty \)
AdS/CFT correspondence
(J.Maldacena hep-th/9711200, review: hep-th/9905111)

large $N_c$, $d=4$, $\mathcal{N}=4$ SYM = IIB strings on $AdS_5 \times S^5$

$\lambda \leftrightarrow \left(\frac{R^2}{\alpha'}\right)^2$ string corrections to SUGRA

$\frac{\lambda}{4\pi N_c} \leftrightarrow g_s$ string loops

$\langle e^{\int h(x)T(x)} \rangle_{\text{field}} = Z_{\text{string}}[g(x, z \to 0) = h(x)]$

$T_{\mu\nu}(x) \leftrightarrow h_{\mu\nu}(x, z \to 0)$

$J_\mu(x) \leftrightarrow A_\mu(x, z \to 0)$

$\text{tr} F^2(x) \leftrightarrow \varphi(x, z \to 0)$

$\vdots$

$\therefore \langle T_{\mu\nu} T_{\alpha\beta} \rangle \sim \frac{\delta^2 \ln Z_{\text{string}}[h]}{\delta h_{\mu\nu} \delta h_{\alpha\beta}} \sim \frac{\delta^2}{\delta h_{\mu\nu} \delta h_{\alpha\beta}} S_{\text{cl}}[h]$

Duality unproven, but many consistency checks performed
How to compute viscosity from AdS/CFT

(G.Policastro, D.Son, A.Starinets hep-th/0104066, hep-th/0205052,
PK, D.Son, A.Starinets hep-th/0405231, PK, A.Starinets hep-th/0506184)

\[ S_{\text{cl}} = \int d^4x \, dz \sqrt{-g} \, g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi = \int d^4x \sqrt{-g} \, g^{zz} \varphi(x, z) \partial_z \varphi(x, z) \left|_{z \to 0} \right. \]

\[ G^{\text{ret}}(k) \sim \frac{1}{z^3} f(k, z) \partial_z f(-k, z) \left|_{z \to 0} \right. \]

\[ f(k, z) \text{ satisfies e.o.m.} \]
\[ f(k, z) \to 1 \text{ as } z \to 0 \]
\[ f(k, z) \text{ outgoing as } z \to z_h \]
\[ \text{(breaks time-reversal invariance)} \]

- For \( G^{\text{ret}}_{\mu\nu, \alpha\beta}(k) \), need linearized metric perturbations \( h_{\rho\lambda} \Rightarrow \text{coupled ODEs} \)
- Shear viscosity \( \eta \sim \langle T_{xy} T_{xy} \rangle \), need \( h_{xy}(x, z) \)
- \( h^y_x = \varphi \) (minimal massless scalar), and decouples (true more generally)
- In the hydro limit \( \omega/T \ll 1, |k|/T \ll 1 \), can solve analytically

Computing real-time correlation functions in strongly coupled SYM amounts to solving a wave equation on a background with BH
Spectral function for stress

\[
\frac{1}{\tilde{\omega}} (\chi(\tilde{\omega}) - \chi^{T=0}(\tilde{\omega})) \left[ \frac{1}{\pi^2 N_c^2 T^4} \right]
\]

\[
\chi(\omega, k) = -2 \text{Im} G^{\text{ret}}_{xy,xy}(\omega, k)
\]

\[
\chi(\omega) \sim \omega, \quad \omega \ll 2\pi T
\]

\[
\chi(\omega) - \chi^{T=0}(\omega) \sim e^{-\gamma\omega}, \quad \omega \gg 2\pi T
\]

\[
\eta = \frac{\pi}{8} N_c^2 T^3
\]

\[
T^3 \text{ by conformal invariance, } N_c^2 \text{ counts d.o.f.}
\]

Spectral function for conserved energy-momentum

\[
\chi_{tx,tx}(\tilde{\omega})
\]

\[
\chi_{tt,tt}(\tilde{\omega})
\]

Hydrodynamic peaks clearly visible in dual classical gravity
Singularities of $G^{\text{ret}}(\omega, k)$

(A.Nunez, A.Starinets hep-th/0302026, PK, A.Starinets hep-th/0506184)

- Infinite series of poles
- $\omega_n = 2\pi n T (\pm 1 - i)$ as $n \to \infty$
- For conserved densities, $\omega_0 \to 0$ as $k \to 0$
- Hydro poles agree with Kubo formula

Singularities of $G^{\text{ret}}(\omega, k)$ are (quasi)normal modes of the dual gravity background
Universality of $\eta/s$

(PK, D.Son, A.Starinets hep-th/0405231, A.Buchel hep-th/0408095)

- Hydro damping rate is set by $\eta/s$
- Shear viscosity $\eta \sim N_c^2$, but $\eta/s$ is finite in the $N_c \to \infty$ limit

At strong coupling:

\[
s = \frac{3}{4} s(\lambda=0) \left[ 1 + O(\lambda^{-3/2}) \right]
\]
\[
\eta/s = \frac{1}{4\pi} \left[ 1 + O(\lambda^{-3/2}) \right]
\]

\[
\frac{\eta}{s} = \frac{1}{4\pi} \text{ does not depend on the dual gravity background} \]

Conformal invariance, SUSY, 3+1 dimensions — not essential for the universality

\[
\therefore \eta/s = \frac{1}{4\pi} \text{ for a large class of strongly coupled field theories}
\]
Speculation: Lower bound on shear viscosity?

η/s ≫ 1 at small coupling

η/s is finite at large coupling

Natural to assume η/s ≥ \( \frac{1}{4\pi} \) in SYM

Is \( \frac{\eta}{s} \geq \frac{1}{4\pi} \) universal?

Prove from first principles?
Generalizations

• Correction $\eta/s = \frac{1}{4\pi}[1 + O(\frac{1}{\lambda^{3/2}})]$ known explicitly in $\mathcal{N}=4$ SYM
  (A.Buchel, J.Liu, A.Starinets \textit{hep-th/0406264})

• Add mass terms to $\mathcal{N}=4$ SYM (destroys scale invariance, reduces SUSY):
  $\eta/s = \frac{1}{4\pi}, \frac{\zeta}{\eta} \approx -4.56(v_s^2 - \frac{1}{3})$
  (P.Benincasa, A.Buchel, A.Starinets \textit{hep-th/0507026})

• Non-zero chemical potential: $\eta/s = \frac{1}{4\pi}$ in $\mathcal{N}=4$ SYM, does not depend on $\frac{\mu}{T}$
Photon production from SYM

(PK, A.Starinets, to appear)

\[
k \frac{d \Gamma}{d k} \propto \frac{k^2 \eta^{\mu \nu}}{e^{k/T} - 1} \text{Im} \Pi^\text{ret}_{\mu \nu}(\omega, k) \Bigg|_{\omega^2 = k^2}
\]

Thick line: \( \frac{x^3}{e^x - 1} \)  
Thin line: \( \frac{x^3}{e^x - 1} f_{\text{SYM}}(x) \)

At small \( x \): \( f_{\text{SYM}}(x) \rightarrow \text{const} \), \( \text{Im} \Pi^\mu_\mu(\omega=k) \sim k \), consistent with hydrodynamics
Summary

• Viscosity can be relatively easily computed for some strongly coupled theories such as $\mathcal{N}=4$ SYM

• Ratio of viscosity to entropy density is $1/4\pi$ for a large class of strongly coupled gauge theories. Lower bound $\eta/s \geq \frac{1}{4\pi}$?

• Limitations of the AdS/CFT approach:
  – SUSY theories, contain scalar fields
  – Conformal in the UV rather than AF
  – Fundamental matter $N_f \ll N_c$
  – $1/\lambda$ corrections doable, $1/N_c$ corrections hard

• Search for universal properties of strongly interacting thermal gauge theories

• Other questions can be addressed in AdS/CFT: energy loss by a heavy quark, existence of resonances, photon production, thermalization...

Can AdS/CFT be useful for heavy-ion physics?